# THE INTERNATIONAL MATHEMATICAL CONGRESS AT TORONTO.

By invitation of the University of Toronto and the Royal Canadian Institute, the International Mathematical Congress was held at the University of Toronto, August 11–16, 1924, in conjunction with the meeting of the British Association for the Advancement of Science which was held August 6–13. Previous congresses had been held as follows: Zurich 1896, Paris 1900, Heidelberg 1904, Rome 1908, Cambridge 1912, Strasbourg 1920. Because many members of the Mathematical Association have made inquiries as to the names of those in attendance and because the proceedings will probably not appear for a year or more, it has seemed desirable to give a fairly full account of the Congress. The list of members of the Congress actually in attendance as here given was furnished by the secretary, Professor J. L. Synge:

ARGENTINA.

C. D. PERRINE, Observ. Nacional, Cordoba.
R. A. VAGO, Arg. Embassy, Washington.

BELGIUM.

A. DEMOULIN, Gand.
L. GODEAUX, Univ. of Brussels.
G. LEMAITRE, Brussels.
E. MERLIN, Gand.
C. SERVAIS, Brussels.
C. DE LA VALLÉE POUSSIN, Univ. of Louvain.

CANADA.

F. D. ADAMS, McGill Univ.
J. R. AMBROISE, Toronto.
D. BUCHANAN, Univ. of Br. Columbia.
C. A. CHANT, Univ. of Toronto.
R. H. COATS, Bur. of Statistics, Ottawa.
H. J. DAWSON, Royal Milit. Coll., Kingston.
A. T. DE LURY, Univ. of Toronto.
L. L. DINES, Univ. of Saskatchewan.
W. P. DOBSON, Hydro-Elec. Power Comm., Toronto.
J. M. DUNCAN, Toronto.
H. B. DWIGHT, Hamilton.
J. D. FERNANDEZ, Toronto.
ALAN FERRIER, Ottawa.
J. C. FIELDS, Univ. of Toronto.
W. FINDLAY, McMaster Univ.
B. A. GILL, Queen's Univ.
A. H. S. GILLSON, McGill Univ.
J. C. GLASHAN, Ottawa.
C. F. GUMMER, Queen's Univ.
C. B. HAMILTON, Toronto.
J. W. HAYWARD, Quebec.
H. P. L. HILLMAN, Hydro-Elec. System, Toronto.
T. H. HOGG, Toronto.
L. HUME, Dept. Natl. Defence, Ottawa.
A. F. HUNTER, Toronto.
T. T. IRVING, Toronto.
L. V. KING, McGill Univ.
H. R. KINGSTON, Univ. of Western Ont.
S. LANZON, Toronto.
A. LEVÉILLÉ, Univ. of Montreal.
G. H. LING, Univ. of Saskatchewan.
C. H. LIGHTHOUSE, Aeronaut. Dept., Ottawa.
N. B. MACLEAN, Univ. of Manitoba.
J. MATHESON, Queen's University.
D. MCARTHUR, Montreal.
W. MCKNIGHT, Tech. Coll., Halifax.
J. MCLEISH, Dept. of Mines, Ottawa.
J. C. MCLENNAN, Univ. of Toronto.
A. R. MCMICHAEL, Toronto.
H. MCTAGGART, Toronto.
N. MILLER, Queen's Univ.
A. MORPHY, Toronto.
D. H. NELLES, Geod. Surv., Ottawa.
A. NEVEILLE, Univ. of Montreal.
N. OGILVIE, Geod. Surv., Ottawa.
J. G. PARKER, Imper. Life Assur. Co., Toronto.
J. PATTERSON, Toronto.
W. J. PATTERSON, Univ. of Western Ont.
J. S. PLASKETT, Dom. Astrophys. Observ., Victoria.
A. POULIOT, Quebec.
L. N. RICHARDSON, Royal Milit. Coll., Kingston.
C. A. ROSE, Am. Nickel Corp., Toronto.
T. R. ROSEBRUGH, Univ. of Toronto.
C. H. RUST, Elec. Light Co., Toronto.
F. SANDERSON, Toronto.
N. A. SHAW, McGill Univ.
R. G. STAGG, Toronto.
A. F. STEVENSON, Toronto.
C. T. SULLIVAN, McGill Univ.
J. L. SYNGE, Univ. of Toronto.
J. J. TRAILL, Hydro-Elec. Co., Toronto.

Miss E. G. Waddell, Univ. of Toronto.
L. A. H. Warren, Univ. of Manitoba.
T. R. Wilkins, Brandon Coll.
H. H. Wolfenden, Grimsby.
W. A. P. Wood, Can. Life Assur. Co., Toronto.
R. B. Young, Toronto.

China.

S. D. Carothers, Hong Kong.

Cuba.

Y. I. Del Corral, Havana.

Czecho-Slovakia.

B. Bydžovský, Univ. of Prag.

Denmark.

T. Bonnesen, Copenhagen.
E. Schou, Copenhagen.
J. F. Steffensen, Hallerup.

France.

Jules Andrade, Univ. of Besancon.
Commandant Barre, Paris.
J. Bosler, Observ. de Marseille.
L. Brillouin, Collège de France.
E. Cartan, Sorbonne.
Gen. V. Charbonnier, Ministr. de la Marine.
J. Chazy, Paris.
V. R. d'Adhemar, Univ. of Lille.
Jules Drach, Paris.
L. Dunoyer, Neuilly.
F. Faure, Dir. de la Rev. polit. et parl.
M. Fréchet, Univ. of Strasbourg.
P. Haag, Clermont-Ferrand.
M. Huber, Paris.
M. Janet, Univ. of Rennes.
G. Koenigs, Sorbonne.
J. Lemoine, Paris.
J. LeRoux, Univ. of Rennes.
A. Levy, Lycée Saint-Louis.
Jean Mascart, Observ., Lyon.
M. Pigeaud, Paris.
Dr. Planiol, École d'Aeronautique, Paris.
J. B. Pomey, Paris.
R. Wurmser, Collège de France.

Georgia.

A. Razmadze, Tiflis.

Great Britain.

Miss E. E. Austin, Bromley.
Arch. Barr, Westerton, Scotland.
T. H. Beare, Univ. of Edinburgh.
A. L. Bowley, Univ. Coll., London.
S. Chapman, Univ. of Manchester.
A. W. Conway, Univ. of Ireland.
P. Cormack, Univ. of Dublin.
A. L. Cortie, Stonyhurst Coll.
J. G. Crowther, London.
A. C. Dixon, Univ. of Belfast.
A. T. Doodson, Tidal Inst., Liverpool.
A. S. Eddington, Cambridge Univ.
R. A. Fisher, Harpenden.
A. P. M. Fleming, Met.-Vickers Elec. Co., London.
A. Fowler, Royal Coll. of Sc.
Henry Fowler, Derby.
I. G. Gibbon, Min. of Health, London.
G. R. Goldsbrough, Sunderland.
A. E. Grasett, Staff Coll., Camberley.
J. G. Gray, Univ. of Glasgow.
G. Greenhill, London.
H. R. Hasse, Univ. of Bristol.
J. B. Henderson, Royal Naval Coll.
G. W. O. Howe, Univ. of Glasgow.
F. R. W. Hunt, Kent.
H. Jeffreys, St. Johns Coll., Cambridge.
C. F. Jenkin, Oxford Univ.
P. A. MacMahon, Cambridge.
E. H. Parker, Glasgow.
W. P. Phelps, London.
H. C. Plummer, Artill. Coll.
A. Robertson, Univ. of Bristol.
L. F. Richardson, Westminster Training Coll.
Ida B. Saxby, Cardiff, S. Wales.
F. J. Selby, Teddington.
Napier Shaw, London.
W. F. Sheppard, Cardona.
D'Arcy Thompson, Univ. of St. Andrews.
F. P. White, St. Johns Coll., Cambridge.
L. Woollard, The Admiralty, London.
W. H. Young, Univ. of Wales.
G. U. Yule, St. Johns Coll., Cambridge.

Holland.

J. A. Barrau, Univ. of Groningen.
W. Kapteyn, Univ. of Utrecht.
W. Van der Woude, Univ. of Leiden.
J. Wolff, Univ. of Utrecht.

India.

G. Prasad, Hindu Univ., Benares.
C. V. Raman, Calcutta.

Italy.

E. Bortolotti, Univ. of Bologna.
M. de Franchis, Univ. of Palermo.
G. Fubini, Univ. of Turin.
F. G. Gianfranceschi, Univ. Gregoriana, Rome.
C. Gini, Univ. of Padua.
G. Giorgi, Inst. of Elec. Engineers, Rome.
G. Muzi, Univ. of Pisa.
G. Peano, Univ. of Turin.

S. PINCHERLE, Univ. of Bologna.
U. PUPPINI, Scuola Ingegneri, Bologna.
F. SEVERI, Univ. degli Studi, Rome.
L. TONELLI, Univ. of Bologna.

NORWAY.

R. BIRKELAND, Univ. of Christiania.
J. BJERKNES, Bergen.
W. BJERKNES, Geophys. Inst., Bergen.
O. ORE, Univ. of Christiania.
C. STØRMER, Univ. of Christiania.

POLAND.

W. SIERPINSKI, Univ. of Warsaw.
S. ZAREMBA, Univ. of Cracovie.

PORTUGAL.

DA COSTA LOBO, Lisbon.
F. DE VASCONCELLOS, Univ. of Lisbon.

ROUMANIA.

G. TZITZEICA, Univ. of Bucarest.

RUSSIA.

M. GUNTHER, St. Petersbourg.
V. A. KOSTITZIN, Moscow.
N. KRYLOFF, Univ. of Kieff.
J. OUSPENSKY, Leningrad.
STEKLOFF, Univ. of Leningrad.
A. V. VASILIEVE, Moscow.

SERVIA.

M. PETROVITCH, Univ. of Belgrade.

SPAIN.

D. J. ALVAREZ-UDE, Madrid.
M. CASTRO BONEL, Univ. of Madrid.
D. TORROJA Y MIRET, Univ. of Barcelona.

SWEDEN.

E. HOLMGREN, Upsala Observ.
J. MALMQUIST, Univ. of Stockholm.
L. E. PHRAGMEN, Kungl. Vetenskapsakad., Stockholm.

SWITZERLAND.

L. J. CRELIER, Univ. of Berne.
H. FEHR, Univ. of Geneva.
R. FUETER, Univ. of Zurich.
M. PLANCHEREL, Univ. of Zurich.

UNITED STATES.

C. R. ADAMS, Brown Univ.
H. L. ALDEN, Univ. of Virginia.
P. L. ALGER, Schenectady, N. Y.
MISS M. ALLEN, Mt. Holyoke Coll.
N. H. ANNING, Univ. of Michigan.
R. C. ARCHIBALD, Brown Univ.
H. T. R. AUDE, Colgate Univ.
M. J. BABB, Univ. of Pennsylvania.
CLARA L. BACON, Goucher Coll.
GRACE M. BAREIS, Ohio State Univ.
L. A. BAUER, Dept. Terr. Magn., Washington.
E. T. BELL, Univ. of Washington.
B. A. BERNSTEIN, Univ. of California.
W. J. BERRY, Brooklyn Poly. Inst.
G. A. BLISS, Univ. of Chicago.
R. L. BORGER, Ohio Univ.
A. BOYAJIAN, Gen. Elec. Co., Pittsfield, Mass.
W. E. BROOKE, Univ. of Minnesota.
F. P. BRACKETT, Pomona Coll.
H. E. BRAY, Rice Inst.
G. BREIT, Dept. Terr. Magn., Washington.
W. C. BRENKE, Univ. of Nebraska.
L. J. BRIGGS, Bur. of Standards, Washington.
R. W. BRINK, Univ. of Minnesota.
E. W. BROWN, Yale Univ.
H. S. BROWN, Hamilton Coll.
E. BUCKINGHAM, Bur. of Standards, Washington.
J. A. BULLARD, U. S. Naval Acad.
W. G. BULLARD, Syracuse Univ.
W. D. CAIRNS, Oberlin Coll.
F. CAJORI, Univ. of California.
G. A. CAMPBELL, Am. T. & T. Co., New York.
I. S. CARROLL, Syracuse.
J. R. CARSON, New York.
C. C. CARTER, Bluffs, Ill.
C. W. CARTER, Am. T. & T. Co., New York.
E. H. CLARKE, Hiram Coll.
G. R. CLEMENTS, U. S. Naval Acad.
A. B. COBLE, Univ. of Illinois.
J. B. COLEMAN, Univ. of South Carolina.
JULIA T. COLPITTS, Iowa State Coll.
LENNIE P. COPELAND, Wellesley Coll.
G. H. CRESSE, Univ. of Arizona.
H. CREW, Northwestern Univ.
LOUISE D. CUMMINGS, Vassar Coll.
H. L. CURTIS, Bur. of Standards, Washington.
D. R. CURTISS, Northwestern Univ.
MARIAN E. DANIELLS, Iowa State Coll.
S. C. DAVISSON, Indiana Univ.
H. M. DADOURIAN, Trinity Coll., Conn.
L. E. DICKSON, Univ. of Chicago.
T. C. DICKSON, Watertown Arsenal.
ELEANOR C. DOAK, Mt. Holyoke Coll.
B. F. DOSTAL, Univ. of Michigan.
A. DRESDEN, Univ. of Wisconsin.
J. A. EIESLAND, West Virginia Univ.
J. S. ELSTON, Travelers Ins. Co., Hartford, Conn.
G. C. EVANS, Rice Inst.
H. S. EVERETT, Bucknell Univ.
H. B. FERNALD, New York.
P. FIELD, Univ. of Michigan.
A. FISHER, Western Union Tel. Co., New York.
J. A. FOBERG, Dept. of Pub. Instr., Harrisburg, Pa.

W. B. FORD, Univ. of Michigan.
T. C. FRY, Am. T. & T. Co., New York.
W. F. FERHARDT, Dayton, Ohio.
J. L. GIBSON, Univ. of Utah.
R. E. GILMAN, Brown Univ.
O. E. GLENN, Univ. of Pennsylvania.
J. W. GLOVER, Univ. of Michigan.
W. S. HAMILTON, Univ. of Wisconsin.
MAY N. HARWOOD, Syracuse Univ.
M. W. HASKELL, Univ. of California.
L. A. HAZELTINE, Stevens Tech. Coll.
OLIVE C. HAZLETT, Mt. Holyoke Coll.
C. M. HEBERT, New York.
N. H. HECK, Coast & Geod. Surv., Washington.
E. R. HEDRICK, So. Branch, Univ. of California.
ROBERT HENDERSON, Equit. Life Assur. Co., New York.
E. HILLE, Princeton Univ.
T. R. HOLLCROFT, Wells Coll.
HELMA L. HOLMES, Univ. of Texas.
ANNA M. HOWE, Newcomb Coll.
GOLDIE P. HORTON, Univ. of Texas.
R. S. HOYT, River Edge, N. J.
D. HULL, Univ. of Notre Dame.
W. J. HUMPHREYS, Weather Bur., Washington.
J. I. HUTCHINSON, Cornell Univ.
EMMA HYDE, Kansas State Agric. Coll.
S. A. JOFFE, Mut. Life Ins. Co., New York.
L. C. KARPINSKI, Univ. of Michigan.
E. H. KENNARD, Cornell Univ.
A. KORZYBSKI, New York.
H. W. KUHN, Ohio State Univ.
ELIZABETH R. LAIRD, Mt. Holyoke Coll.
P. A. LAMBERT, Lehigh Univ.
A. E. LANDRY, Catholic Univ. of Amer.
D. D. LAUN, Univ. of Chicago.
ELIZABETH LESTOURGEON, Univ. of Kentucky.
H. C. LEVINSON, Chicago.
MAYME I. LOGSDON, Univ. of Chicago.
L. A. MCCOLL, Western Elec. Co., New York.
C. C. MACDUFFEE, Ohio State Univ.
W. MARSHALL, Purdue Univ.
H. H. MARVIN, Univ. of Nebraska.
T. E. MASON, Purdue Univ.
S. J. MAUCHLY, Dept. Terr. Magn., Washington.
G. A. MILLER, Univ. of Illinois.
G. R. MIRICK, Lincoln School, New York.
EUGENIE M. MORENUS, Sweet Briar Coll.
F. MORLEY, Johns Hopkins Univ.
E. C. MOLINA, Am. T. & T. Co., New York.
C. N. MOORE, Univ. of Cincinnati.
M. MORRIS, Case School of Appl. Sc.
H. M. MORSE, Cornell Univ.
E. J. MOULTON, Northwestern Univ.
F. R. MOULTON, Univ. of Chicago.
J. R. MUSSELMAN, Johns Hopkins Univ.
A. MCADIE, Readville, Mass.
G. F. MCEWEN, Univ. of California.
J. MCGIFFERT, Rensselaer Poly. Inst.
J. J. NASSAU, Case School of Appl. Sc.
EDNA P. PEPPER, Chicago.
T. A. PIERCE, Univ. of Nebraska.
J. PIERPONT, Yale Univ.
E. A. PORTER, Indianapolis.
V. E. POUND, Univ. of Buffalo.
L. J. REED, Johns Hopkins Univ.
J. N. RICE, Catholic Univ. of Amer.
R. G. D. RICHARDSON, Brown Univ.
G. M. ROBISON, Trinity Coll., N. C.
C. K. ROBBINS, Purdue Univ.
E. D. ROE, JR., Syracuse Univ.
MRS. E. D. ROE, Syracuse, N. Y.
W. H. ROEVER, Washington Univ.
J. H. ROGERS, Univ. of Missouri.
J. ROSENBAUM, Milford, Conn.
D. A. ROTHROCK, Indiana Univ.
F. H. SAFFORD, Univ. of Pennsylvania.
IDA M. SCHOTTENFELS, Chicago.
W. H. SHERK, Univ. of Buffalo.
W. A. SHEWHART, Western Elec. Co., New York.
J. A. G. SHIRK, Kansas State Tea. Coll.
J. A. SHOHAT, Univ. of Chicago.
L. SILBERSTEIN, Eastman Kodak Co.
W. G. SIMON, Western Reserve Univ.
T. MCN. SIMPSON, Randolph-Macon Coll.
T. M. SIMPSON, Univ. of Florida.
MARY EMILY SINCLAIR, Oberlin Coll.
C. H. SISAM, Colorado Coll.
E. B. SKINNER, Univ. of Wisconsin.
C. S. SLICHTER, Univ. of Wisconsin.
H. L. SLOBIN, Univ. of New Hampshire.
A. W. SMITH, Colgate Univ.
CLARA E. SMITH, Wellesley Coll.
HELEN SMITH, Iowa State Coll.
SARAH E. SMITH, Mt. Holyoke Coll.
VIRGIL SNYDER, Cornell Univ.
MAY J. SPERRY, Syracuse Univ.
C. E. ST. JOHN, Mt. Wilson Observ.
W. F. G. SWANN, Univ. of Chicago.
K. D. SWARTZEL, Univ. of Pittsburgh.
J. S. TAYLOR, Mass. Inst. of Tech.
E. M. THOMAS, Boston.
W. M. THORNTON, Univ. of Virginia.
M. O. TRIPP, Wittenberg Coll.
W. H. TSCHAPPAT, Ord. Dept., Washington.
BIRD M. TURNER, West Virginia Univ.
A. L. UNDERHILL, Univ. of Minnesota.
G. VAN BIESBROECK, Yerkes Observ.
H. S. VANDIVER, Cornell Univ.
J. A. L. WADDELL, New York.
J. H. WEAVER, Ohio State Univ.
A. H. WHEELER, Worcester, Mass.
H. S. WHITE, Vassar Coll.
W. WHITED, Pa. State Highway Dept.
A. W. WHITNEY, Natl. Bur. of Cas. & Surety Underwr., New York.
R. L. WILDER, Univ. of Texas.

W. L. G. WILLIAMS, Cornell Univ.
W. F. WILLCOX, Cornell Univ.
E. B. WILSON, Harvard Univ.
ELIZABETH W. WILSON, Washington.
RUTH G. WOOD, Smith Coll.
E. W. WOOLARD, Weather Bur., Washington.
B. F. YANNEY, Coll. of Wooster.
C. H. YEATON, Oberlin Coll.
O. J. ZOBEL, New York.

At the opening session in Convocation Hall on Monday morning addresses of welcome were given by Minister H. S. Beland, M.D., in English and French on behalf of the Dominion government, and by President Robert Falconer on behalf of the University. Addresses followed by Dr. J. C. Fields, Chairman of the Organizing Committee, who was elected president of the Congress, and by M. de la Vallée Poussin, president of the International Mathematical Union. The list of delegates from various sections, societies, colleges and universities was read by Professor Koenigs, the general secretary of the Union. A group photograph of the Congress was taken at noon and was placed on sale later in the week by the University of Toronto Press.

Monday afternoon and Tuesday, Wednesday, Friday and Saturday mornings were devoted to sessions for reading contributed papers in the following sections: I, Algebra, theory of numbers, analysis; II, Geometry; III (*a*), Mechanics, mathematical physics; III (*b*), Astronomy, geophysics; IV (*a*), Electrical, mechanical, civil and mining engineering; IV (*b*), Aeronautics, naval architecture, ballistics, radio-telegraphy; V, Statistics, actuarial science, economics; VI, History, philosophy, didactics. The number of papers read in these various sections were respectively 51, 38, 24, 11, 28, 14, 23, 13, 15 other papers being given before III (*a*) and III (*b*) jointly, and 4 others before IV (*a*) and IV (*b*) jointly. Excellent abstracts of the majority of papers were distributed at the time of registration; these are still available to a limited extent through the University of Toronto Press. Besides these, eight general lectures were given at other times during the week:

C. Størmer, "Modern Norwegian researches on the aurora borealis;"

F. Severi, "Géométrie algébrique;"

E. Cartan, "La théorie des groupes et les recherches récentes de géométrie différentielle;"

W. H. Young, "Some characteristic features of twentieth century pure mathematical research;"

L. E. Dickson, "Outline of the theory to date of the arithmetics of algebras;"

S. Pincherle, "Opérations fonctionelles;"

J. LeRoux, "Sur l'intégration des équations aux derivées partielles par des intégrales définies;"

J. Pierpont, "Non-Euclidean geometry from a non-projective standpoint."

It will be of interest to all mathematicians of America to note that Professor Fueter of the Editorial Committee for the works of Euler announced that the cost of printing these is so enhanced that only by obtaining an increased number of subscriptions will it be possible to complete the enterprise with any degree of promptness.

The business meeting of the International Mathematical Union held on

Friday morning was of particular interest to all mathematicians, inasmuch as this is the only formal bond uniting mathematicians of different nationalities. By ballot of the delegates from the "sections" of the various nationalities (*e.g.*, the five delegates from the American Section) Professor Pincherle was elected president of the Union for the next four years, and Professors Bliss, Fehr and Phragmen were elected vice-presidents. As the result of many informal discussions carried on between the members of the various sections, a resolution by the American Section was received by the present officers of the Union for transmission to the International Research Council which meets next July for the transaction of business. The resolution raised the question whether it is the proper time for the revision of the condition of admission to the Union. (It was understood that the chief point of contention would be with regard to the admission of German mathematicians.) Unanimous agreement with this action of the American Section was announced by the delegations from Italy, Holland, Sweden, Denmark, Norway and England. While there were distinct differences of judgment and of sentiment in this question, it was a matter for mutual congratulations that a common ground was reached in the same fine spirit which during these very days characterized the conference of European and American statesmen at London. Professors Archibald, Bortolotti, Fréchet, van der Woude, and W. H. Young were appointed as an *International Committee on Bibliography*.

Many social events were provided for those attending the Congress and the meeting of the British Association. An organ recital was given Sunday evening by the university organist, Dr. F. A. Mouré. A garden party was given by Professor J. C. MacLennan of the department of physics at York Club Monday afternoon and a garden party by His Honour the Lieutenant-Governor of Ontario at Government House Tuesday afternoon. On Tuesday evening the University and Royal Canadian Institute held a Conversazione at Hart House; in various parts of this wonderful social center occurred a clever Canadian play, "Brothers in Arms," by Merrill Denison, a display of fencing under the direction of a Canadian fencing champion, a base-ball game, scientific "lecturettes" by three members of the British Association, playing by bagpipers and music for dancing. On Wednesday afternoon the degree of doctor of science was conferred by the University on Professors Koenigs, Severi, Stekloff and de la Vallée Poussin, and on four representatives of the British Association; after this convocation came a garden party by the University, a cricket match on the campus and an excursion around the city in motor charabancs furnished by the University. Thursday was given over to various group outings, chiefly to a steamer excursion to Queenston where the mammoth generating units of the Hydro-Electric Power Plant were shown in great detail by employes, and to Niagara Falls where the visiting scientists were the guests of the Hydro-Electric Power Commission at a bountiful luncheon. A garden party given at the Grange by the Council of the Art Gallery Friday afternoon, a soirée at the Toronto Hunt Club Friday evening and a dinner for the Congress at Hart House Saturday evening gave a

finish to a week of festivities highly appreciated by all in attendance. Sums contributed by the members of the Congress provided for a wreath deposited Saturday afternoon at the memorial to the University's soldier dead and for a permanent tablet which is to be placed thereon.

W. D. CAIRNS.

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## THE MATHEMATICAL ASSOCIATION.

The Trustees of the Association have approved of the organization of a Nebraska Section. Account of the organization meeting appeared in the October number of the MONTHLY.

The following forty-eight individuals and eight institutions, on applications duly certified, have been elected to membership in the Association:

*To Individual Membership.*

HARRIET ANDERSON, B.S. (Iowa State Coll.). Prof., Grand Island Coll., Grand Island, Nebr.
ETHEL L. ANDERTON, A.M. (Yale). Instr., Wellesley Coll., Wellesley, Mass.
MAURICE BACKER, A.B. (Westminster). Henderson, Ky.
L. F. BENSON, B.S. (East Texas State Teachers Coll.). Forney, Texas.
W. W. BIGELOW, A.B. (Beloit). Computer, Coast & Geodetic Surv., Washington, D. C.
GREGORY BREIT, Ph.D. (Johns Hopkins). Dept. of Terrestrial Magnetism, Washington, D.C. *36th St. and Broad Branch Rd.*
MARY A. CAMPBELL, A.M. (Texas). Head of dept. of math., South Park Jr. Coll., Beaumont, Texas.
JULIA R. S. CHELLBORG, B.S. (Hunter). Asst. prof., Hunter Coll., New York, N. Y.
RUFUS CRANE, A.B. (Middlebury), B.S. (Mass. Inst. of Tech.). Asst. prof. of eng., Ohio Wesleyan Univ., Delaware, Ohio.
C. W. R. CRUM, M.D. Brunswick, Md.
L. A. DEESZ, B.S. (Carnegie Inst.). Asst. supt. of elec. dept., Colo. Fuel and Iron Co., Pueblo, Colo.
BERTHA K. DUNCAN, A.M. (Texas). Head of dept. of math., Grenada Coll., Grenada, Miss.
W. M. EWING. Junior student, Rice Inst., Houston, Texas.
J. M. HARTSFIELD. Junior student, Rice Inst., Houston, Texas.
(Recommended by Professors Rice and Ford.)
ANNIE W. FLEMING, A.M. (California). Asst. prof., Iowa State Coll., Ames, Ia.
J. F. FOX. Mathematician, Coast & Geodetic Surv., Washington, D. C.
ALICE A. GRANT, A.B. (Toronto; McMaster). 238 Gano St., Providence, R. I.
F. C. HENROTEAU, Dr. of physics and math. (Brussels). Head of astrophysical dept., Dominion Observ., Ottawa, Can.
A. L. HILL, A.B. (Doane). Head of dept. of math., Peru, Nebr.
ELINOR V. HOLLIS, M.S. (Chicago). 11 Boynton St., Worcester, Mass.
H. K. HUGHES, A.B. (Iowa). Grad. asst., Univ. of Iowa, Iowa City, Ia.
J. F. HULSE, A.B. (Coll. of Ozarks). 705 W. 22½ St., Austin, Texas.
MABEL HUTCHINS, A.B. (Blue Mountain). Head of dept. of math., Blue Mountain Coll., Blue Mountain, Miss.
MARY F. JACKSON, B.S. (Nebraska). Teacher, High School, Lincoln, Nebr.
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## ANNUAL MEETING OF THE TEXAS SECTION.

The third annual meeting of the Texas section of the Mathematical Association of America was held in conjunction with the mathematics section of the Texas State Teachers Association at Ft. Worth, Texas, on Friday morning and afternoon, December 1, 1923. Dr. C. N. Wunder, dean of Southwestern University, Georgetown, Texas, and professor of mathematics, presided over the morning session as chairman of the mathematics section of the Texas State Teachers Association. The afternoon session was presided over by Professor L. R. Ford, of Rice Institute, Houston, Texas, chairman of the Texas section of the Association.

There were fifty-five (55) present, including the following members of the Association:

J. M. Bledsoe, Myrtle C. Brown, J. E. Burnam, J. W. Calhoun, J. P. Downer, H. J. Ettlinger, G. C. Evans, L. R. Ford, A. J. Hargett, W. A. Nelson, Hugh Porter, E. R. Tucker, C. N. Wunder.

The following papers were presented:

(1) "Some interesting theorems of elementary geometry" by Professor J. M. BLEDSOE, East Texas State Teachers College;

(2) "The esthetic element in mathematics" by Professor J. W. CALHOUN, University of Texas;

(3) "Plane geometry constructions by means of the compasses alone" by Professor H. J. ETTLINGER, University of Texas;

(4) "Numerical integration of differential equations" by Professor L. R. FORD, Rice Institute;

(5) "Some further applications of Duhamel's theorem including integration of a non-uniformly convergent series termwise" by Professor H. J. ETTLINGER, University of Texas;

(6) "Mathematics of economics" by Professor G. C. EVANS, Rice Institute.

The present officers were reëlected to serve for the next year. It was voted to instruct the secretary to ascertain the sentiment of the members by a mail questionnaire on the proposition of changing the time of meeting, so as to have a separate meeting at Christmas time.

H. J. ETTLINGER, *Secretary-Treasurer.*

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## ON THE INTEGRABILITY OF A CONTINUOUS FUNCTION.

By H. J. ETTLINGER, University of Texas.

In a recent paper [1] the writer gave a proof of the integrability of a continuous function, based on Moore's form of Duhamel's theorem.[2] It is the object of this paper to give a proof of what is essentially Moore's form of Duhamel's theorem from which the integrability of a continuous function is obtained as a direct corollary. As in the former paper, use is made of a limit theorem on areas.[3] The present proof gives a principle for obtaining an outer sum which is a monotonically decreasing function of $n$, which in simplicity leaves nothing to be desired.

We will now state the principal theorem.

FUNDAMENTAL THEOREM ON SUMMATION: *Let the interval* $I : a \leqq x \leqq b$ *be divided into n subdivisions in any manner whatsoever, each of length* $\Delta x_i = \Delta x_i(n)$, *such that if P is any fixed point of I whose abscissa is* $x_P$, *and* $\Delta x_P = \Delta x_P(n)$ *is a subdivision which contains P for every value of n, then* $\lim n \to \infty \ \Delta x_P = 0$. *Let* $H_i = H_i(n)$ *be any set of numbers corresponding to* $\Delta x_i$, *such that* (1) $|H_i| \leqq K$

[1] H. J. Ettlinger, "A Simple Form of Duhamel's Theorem and Some New Applications," this MONTHLY (*1922*, 239–250).

[2] R. L. Moore, "On Duhamel's Theorem," *Annals of Mathematics*, second series, vol. 13, 1912, pp. 161–166. For other forms of this theorem and substitutes thereof, see the preceding reference, this MONTHLY (*1922*, 240–243).

[3] *L. c.*, this MONTHLY (*1922*, 243). See also H. J. Ettlinger, "An Elementary Proof of a Fundamental Lemma Concerning the Limit of a Sum," *Bulletin of the American Mathematical Society*, 1923, vol. 29, pp. 219–223.

*for every value of* $i$ $(\leqq n)$ *and* $n$, *where* $K$ *is a positive constant, and* (2) *if* $H_P = H_P(n)$ *is that number which corresponds to* $\Delta x_P$, *let* $\lim n \to \infty\ H_P = f(x_P)$, *where* $f(x)$ *is a single-valued continuous function over* $I$. *Then*

$$\lim_{n \to \infty} \sum_{1}^{n} H_i \Delta x_i \text{ exists.}$$

PROOF: Designate the points of division of $I$ by $x_i = x_i(n)$, $(i = 1, 2, \cdots, n-1)$, the internal points of division, together with the end points $x_0 = a$ and $x_n = b$. Now for any particular value of $n$ consider all the interior points of division $x_1(2)$; $x_1(3)$, $x_2(3)$; $x_1(4)$, $x_2(4)$, $x_3(4)$; $x_1(5)$, $x_2(5)$, $x_3(5)$, $x_4(5)$; $\cdots$; $x_1(n)$, $x_2(n)$, $\cdots$, $x_{n-1}(n)$. For any particular value of $n$ let the total number of these points be $p = p(n)$. It is clear that $p$ will be at least as great as $n - 1$, since although any point of a given set $x_i(s)$ may coincide with a point of a set previously chosen $x_j(r)$, where $r < s$, none of the points $x_i(n)$, that is, the last $n - 1$ points chosen, may coincide with each other. The totality of the pairs of consecutive internal points of division, together with $a$ and $b$, divide $I$ into $p + 1$ subdivisions which we shall call $\delta_i = \delta_i(n)$.

Let $Y_i = Y_i(n)$ be the largest value of $f(x)$ in $\delta_i$, and $M$ and $m$ the largest value and smallest value respectively of $f(x)$ in $I$. Form the outer sums

$$\begin{aligned} \overline{A}(1) &= Y_1\delta_1 = M(b-a), \\ \overline{A}(2) &= Y_1\delta_1 + Y_2\delta_2, \\ &\cdots\cdots\cdots \\ \overline{A}(n) &= Y_1\delta_1 + Y_2\delta_2 + \cdots + Y_{p+1}\delta_{p+1} = \sum_{1}^{p+1} Y_i\delta_i, \end{aligned} \tag{1}$$

where on the right side of the equation, the dependence of the numbers on $n$ is not explicitly stated for brevity's sake. Now the outer sum $\overline{A}(n)$ is a monotonic decreasing function of $n$, for if the number of subdivisions is increased from $n$ to $n + 1$, any subdivision $\delta_i(n)$ will either remain unchanged, *i.e.*, unsubdivided, or it will be divided into several parts; but in any case the maximum value of $f(x)$ in any one of these new subdivisions of $\delta_i$ cannot be greater than $Y_i(n)$. Now for all values of $n$

$$\overline{A}(n) \geqq m(b-a).$$

Hence by a well-known limit theorem $\overline{A}(n)$ will approach a limit, $A$, as $n \to \infty$. It is to be noticed that since $p \geqq n - 1$, $p$ will increase without limit as $n \to \infty$. The limit $A$ is the *upper Darboux integral*, whether $f(x)$ be continuous or not, provided $f(x)$ is bounded.

Now let $\Delta x_i = \Delta x_i(n) = x_i(n) - x_{i-1}(n)$ and form the sum

$$\sum_{1}^{n} H_i \Delta x_i. \tag{2}$$

We may compare this sum with (1) if we properly correlate the $\delta_i(n)$ of (1) with the corresponding $\Delta x_i(n)$ of which it is a part and choose the same $H_i(n)$

for all those $\delta_i(n)$'s which are part of the same $\Delta x_i(n)$. If we take the difference between (2) and (1) written in terms of $\delta_i$'s, we have

$$\sum_1^{p+1} [Y_i - H_i]\delta_i. \tag{3}$$

If $\delta_P = \delta_P(n)$ is that subdivision which for each value of $n$ contains the point $P$ where $P$ is any fixed point of $I$ whose abscissa is $x_P$, and $Y_P = Y_P(n)$ and $H_P = H_P(n)$ the corresponding values of $Y_i(n)$ and $H_i(n)$ respectively, then at the same time that $n \to \infty$, $\delta_P \to 0$, $Y_P \to f(x_P)$ and $H_P \to f(x_P)$ since $\Delta x_P \to 0$ and $f(x)$ is continuous. We note also that the numerical value of the difference between $Y_i$ and $H_i$ is never greater than the larger of the two numbers $|K - |M||$, $|K - |m||$. Hence we may apply the lemma on the limit of such a sum [1] and obtain

$$\lim_{p\to\infty} \sum_1^{p+1} H_i\delta_i = \lim_{n\to\infty} \sum_1^n H_i\Delta x_i = A.$$

Hence the Fundamental Theorem on Summation is proved.

We may choose as a particular set of numbers $H_i$, the values of $f(x)$, $f(t_i)$, where $t_i = t_i(n)$ is any abscissa in $\Delta x_i(n)$, for if $t_P$ is the abscissa corresponding to $\Delta x_P$, then since $f(x)$ is continuous we have lim $n \to \infty$ $f(t_P) = f(x_P)$ and $f(t_i)$ is never greater than the larger of the two numbers $|M|$, $|m|$. Hence

*Corollary* I. *$f(x)$ is integrable over $I$.* We will remark that it can be shown that the limit $A$ is the same for two different modes of subdivision, by considering the totality of intervals defined by each pair of consecutive points of the total number of points of both subdivisions and choosing the values of $f(x)$ on these intervals in accordance with the principle of choice of $H_i$ in the proof of the theorem above. Hence $A = \int_a^b f(x)dx$, and we have

*Corollary* II.

$$\lim_{n\to\infty} \sum_1^n H_i\Delta x_i = \int_a^b f(x)dx.$$

Finally we will note that if in the original theorem $y_i = y_i(n)$, where $y_i(n)$ is the smallest value of $f(x)$ in $\delta_i(n)$, be chosen for each $\delta_i(n)$, we obtain a monotonically increasing inner sum by means of which it is readily proved that the *lower Darboux integral* exists, provided $f(x)$ is bounded.[2]

[1] H. J. Ettlinger, *Bulletin of the American Mathematical Society*, *l. c.*, p. 219; *cf.* remark on unequal subdivisions on p. 221, preceding case 2.

[2] It may be readily seen from Moore's form of Duhamel's theorem (*l. c.*) that the upper and lower Darboux integrals are equal if $f(x)$ is continuous "almost everywhere" over $I$, that is, except for a null set. The above argument also lends itself readily to a very simple proof that if $f(x)$ is *measurable* over $I$, it is integrable over $I$ in the sense of Lebesgue. The details of such a proof have been carried through by W. L. Ayres, a graduate student at the University of Texas.

# THE HISTORY OF MODERN CALCULATING MACHINES, AN AMERICAN CONTRIBUTION.

By L. LELAND LOCKE.

During the past two decades the calculating machine has been developed and commercialized to such an extent that it may be said to rival logarithms in importance as a labor-saving device. Coming as the climax of three centuries of somewhat unproductive experimentation, its perfecting lacks the picturesqueness which belongs to the invention of logarithms. Just how soon the latter device will be remembered only as a curiosity in the development of mathematics it is difficult to predict. Certain it is that the calculating machine has not attracted the attention of the mathematician to the extent it deserves, witness the complete absence of literature on the subject in American technical journals and an almost equal void in foreign publications.

The available information concerning such machines is chiefly to be found in patents, descriptive articles on the mechanical features of particular machines, catalogs of collections and exhibitions, advertising material (sometimes with historical notes of more or less value), and a few general treatises. Moreover, these various elements, with diverse and often competing aims, have produced an ambiguous terminology which is a source of dissatisfaction to the careful reader. Inaccuracies in statement, once made, have been repeated, and incorrect dating, particularly where priority is involved, has become a serious fault. The first of the following notes is merely suggestive in the matter of nomenclature in laying a foundation for the second note, which is a compilation of facts bearing on one of the more important questions of priority.

**1. Terminology.** Mathematics is a science which has slowly evolved with the culture of the race and from the needs of everyday life, the concepts and terms being the result of a long-continued process of refinement. The designing of calculating machines is an art which has been derived from the science itself and it has the disadvantage of possessing a terminology appropriated from and based on that of mathematics, this terminology occasionally being none too well defined. It is but natural that the inventor should choose the most high-sounding phrases to describe his work, a practice scarcely so serious in its effects as the very prevalent tendency of some manufacturers to endeavor to make a machine which is designed for a particular purpose function in every conceivable situation. The term "calculating machine" is often applied to a mechanism which has no mechanical capacity for carrying out the four processes of addition, subtraction, multiplication, and division. Such machines were primarily designed for one or more of these four fundamental operations, the remaining processes being worked out by means of applied formulas. For example, any adding machine may be used for calculations involving subtraction, multiplication, and division, as it is possible to work out all of these through the medium

of addition, with pencil and paper as well as with a machine. In order to be mathematically correct, however, the name *Calculating Machine* should be reserved for such machines as have the mechanical capacity for working out each of the four operations by a direct mathematical method.

No serious attempt has yet been made to standardize this terminology although Lenz [1] has given a brief statement of the practice used in the German Patent Office. For the purpose of the next note the following convenient terminology is suggested.

*Machine.* Consider a decimal scale, where nine units have been registered in any order. If, when the tenth unit is registered in this order, provision is made automatically to carry one to the next higher order, this mechanical feature may be said to transform the device into a *machine.* The Japanese soroban is probably the most versatile calculating device ever created but lacks the one feature essential to classify it as a machine.

*Counting Machine.* A machine designed to receive entries in the units' or ones' order only, the register in the higher orders being the accumulations from the carry, may be called a *counting machine.* Such a machine is exemplified in any one of the various types of meters.

*Adding Machine.* A machine designed to receive entries in all orders, successively or simultaneously, may be called an *adding machine.*

*Calculating Machine.* Two additional features will be deemed necessary to produce a calculating machine: (1) a carriage by which the numeral wheels or registering dials and the selector mechanism may have their relative positions shifted; (2) provision for the performance of subtraction directly. By "directly" is meant that the number to be subtracted is entered on the set-up and combined with the number recorded in the machine, either by a reversal of the numeral wheels or by mechanically produced over-addition. In subtraction by over-addition the difference between the subtrahend and the next higher power of ten is added to the minuend. The 1 from the carry is taken care of by inserting a succession of 9's or by providing a cut-off for the carry at the proper point. In an article on "Calculating Machine Mathematics" in *The Mathematics Teacher,* XV, 7, the use of the word *supplement* was suggested for the difference between a number and infinity, infinity being here defined as a power of ten which will carry the 1 off the machine. An earlier use of the word in a similar sense by Barr in certain British patents has since been noted.

For descriptive purposes the parts of the common calculating machine, with their functions, may be named as follows:

*Set-up Mechanism.* The device by which a given number is entered on the machine will be designated the *set-up mechanism.* The older form of set-up was usually a slide, either a straight line or an arc of a circle. This form is now being rapidly replaced by the keyboard. The set-up mechanism is sometimes called the *installing mechanism,* a somewhat more desirable term.

*Selector Mechanism.* The mechanism which selects the proper mechanical

[1] K. Lenz: *Die Rechenmaschinen und das Maschinenrechnen,* Leipzig, 1915, p. 22.

movements to correspond to the number set up is designated as the *selector* or *differential mechanism*. It may well be called the brains of the machine, the design of which determines one of the commoner classifications of machines. The Leibniz-Thomas stepped cylinder is shown in fig. 2 and the Baldwin cam-operated radial pins in fig. 3. Among other types may be mentioned the rack-and-pinion of the Millionaire and Mercedes machines, the mechanical Pythagorean table of the Bollée and the Millionaire, and the squirrel cage or lantern wheel of the Monroe.

*Registering Mechanism.* The set of circular or cylindrical dials or numeral wheels upon which results are usually recorded is called the *registering mechanism*.

*Carry Mechanism.* The function of the carry mechanism is fully implied in the term.

*Control Mechanism.* All devices for the prevention of inaccurate operation, mechanical or human, such as checks for overrotation, locking devices and stops, will be included under the term *control mechanism*.

*Erasing Mechanism.* The various types of devices used to return the numeral dials to the zero position are known as *erasing* or *zero-setting mechanisms*.

**2. A new classification of calculating machines.** The classifications heretofore used have been either on a basis of function, as adding, listing, multiplying, dividing, etc., or of the employment of essential mechanical elements, as the stepped cylinder, cam-operated radial pins, mechanical Pythagorean table, etc. The new classification is based on the sequence of operations as follows:

Type I. Monophase machines,
Type II. Non-reversible cycle machines,
Type III. Reversible cycle machines.

In type I the carry action occurs simultaneously with the digital selection or registration. Such machines may be unidirectional or reversible, the latter type not having been developed commercially. In type II the digital registration and the carrying of tens occur in cyclic order. The cycle is divided into two parts, the first part serving for digital registration, and the second part for the carrying of the tens from the lower to the higher numeral wheels. Obviously this cyclic action is not reversible and in order to reverse from positive to negative numeral

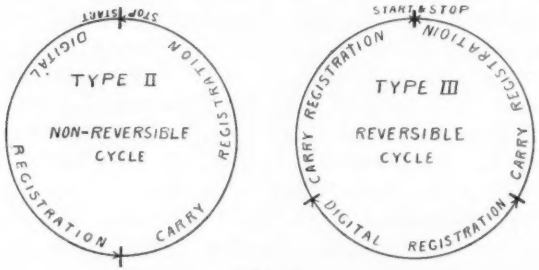


FIG. 1.

wheel registrations, special reversing mechanism must be installed which will not reverse the cycle. Such mechanism may be manually or automatically controlled. In type III the digital registration and the carrying of the tens occur in reversible cyclic order. The zone of digital registration is located midway in the cycle, with a carrying zone provided on each side. The carry action will therefore function after the digital registration in either direction of operation. In this type no special reversing mechanism is required, merely a reversal of direction of the driving shaft which operates the machine. Figure 1 shows the two latter types in diagrammatic form.

**3. The Baldwin-Odhner priority claims.** To Pascal belongs the honor of having conceived and constructed the first calculating machine of which certain models and a description have been preserved. Of more than fifty models constructed by Pascal, several are extant, the oldest of date 1642. Following the classification above, this would be called an adding machine. However it is interesting to note Pascal's genius in discovering the process of subtraction by overaddition, very similar to the method commonly employed with keydrive machines. Of the many attempts following the work of Pascal, only those in direct sequence in the development of the commercial calculating machine of today will be mentioned. The first of these was the multiplying machine of Leibniz, who designed the stepped cylinder as a selector mechanism, which became the distinctive feature of one of the two great classes of commercial calculating machines.

The next real advance was due to Thomas de Colmar, who utilized the stepped cylinder of Leibniz in a machine built in 1820, although it is said that he was not familiar with the work of the latter. The Thomas invention is the prototype of practically all commercial machines built before 1875, and of a goodly share of those developed since that time. In fig. 2, *A* is the stepped cylinder, on the

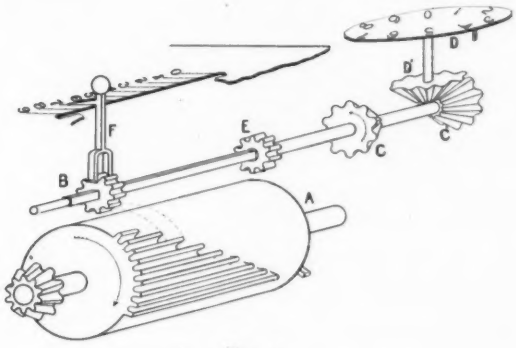


FIG. 2.

surface of which are nine cogs of length varying from one to nine units. This cylinder always rotates in one direction as indicated by the arrow. On the

shaft above and parallel to the cylinder a pinion $B$ slides to a position indicated by a scale on the edge of the groove in the upper plate. When the pointer on handle $F$ indicates 7, the pinion will engage with the seven longest cogs on the cylinder and miss the eighth.

The motion is transmitted by the shaft to the pair of bevel gears $C$ and $C'$. The dial $D$ with a bevel gear $D'$ on the lower end of its shaft is mounted on a carriage. By means of a slide lever this bevel gear may be placed in mesh with either $C$ or $C'$ at will, one of which rotates it forwardly for addition, the other rotating it reversely for subtraction. Thus the Thomas machine is transformed by a lever shift from an adding and multiplying machine into one adapted to subtraction and division.

Frank Stephen Baldwin, on September 8, 1873, applied for and on February 2, 1875, was granted a patent on the first [1] practical calculating machine which at all times had the capacity to add, subtract, multiply, and divide with no resetting of the mechanism and with no form of conversion for any of the processes. It is the purpose of this note to describe the machine briefly and to submit evidence as to the validity of the two qualifying adjectives, *first* and *practical*, the former of which has frequently by implication been denied.

*Description.* In the Baldwin machine the stepped cylinders of Leibniz and Thomas are replaced by a single cylinder. The selector mechanism is a series of sets of nine radially placed pins, one set for each decimal order and located in planes at right angles to the axis of the cylinder. By means of a rotatable cam these pins may be made at will to project beyond the surface of the cylinder as shown in fig. 3. In this figure the handle which operates the cam is located

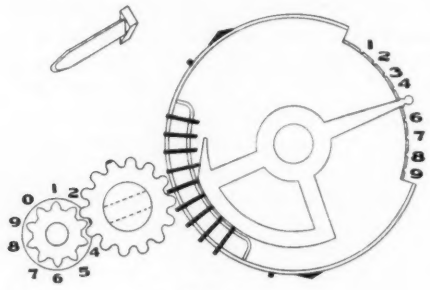


FIG. 3.

at 5, corresponding to which five of the pins project beyond the surface and thus engage cogs of the intermediate pinion, which in turn is in mesh with the recording or numeral dials. The five pins are said to be active and the four which do not

[1] Possibly the nearest approach to such a machine was that of Stanhope (1777), which did not pass the experimental stage. This invention was based on certain mechanical principles inherently weak which would seem to preclude its development into a practical machine without radical changes in design.

project are inactive. This set of active and inactive pins[1] controlled by a rotatable cam in the plane of the pins is the "active and inactive pin principle," which is the distinctive feature of the second great type of calculating machine.

A second feature of the Baldwin machine is the "wedge" shown in figure 3, which passes through a slot (indicated by dotted lines) in the shaft of the intermediate gear. When the numeral wheel passes from 9 to 0 or from 0 to 9 a stud or post on the dial pushes the slide through the shaft toward the large cylinder as shown in figure 4. The bevel edge on the larger end of the wedge throws a

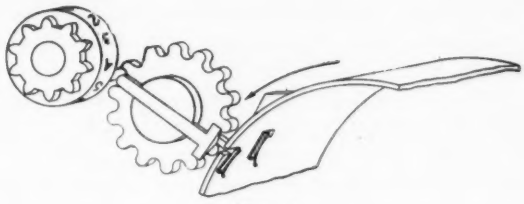

FIG. 4.

finger on the surface of the cylinder into the path of the pinion which operates the next numeral wheel and 1 is carried to that wheel. The large cylinder or set-up barrel may be moved to the right or left to bring the pins opposite any order of the dials or numeral wheels, which are mounted in the frame of the machine.

*Chronology.* On October 5, 1872, Mr. Baldwin filed with the United States Patent Office a caveat, attested on September 28, 1872, containing complete drawings and description of this machine. On September 8, 1873, application was filed for a patent which was granted on February 2, 1875, as U. S. No. 159244. A complete model of the machine was submitted with the application, and is now in the possession of the Smithsonian Institution. The design of this machine was therefore, by documentary evidence, completed before September 28, 1872.

*Practicability.* Of the first ten machines built, eight were sold as follows: Pennsylvania R.R., Philadelphia, (1); Fairman Rogers, Philadelphia, (1); State Insurance Department, Albany, (1); Insurance Companies, New York, (3); War Department, Washington, (1); Interior Department, Washington, (1).

All of the above were sold before 1876. The pertinent question here is, were these machines in the experimental stage, or were they calculating machines. This question is fully answered in the one letter which space allows.

[1] The cam operated radial pin was utilized in a machine designed by Roth in 1841. The disposition of the parts and general design in no way resemble the work of Baldwin. This machine was never developed commercially and was unknown to Baldwin and probably to Odhner, so that it may safely be said to have had no influence in the development of this type of machine.

The works of inventors whose machines were not developed to a commercial stage are milestones in the pathway of progress and should not be subject to disparagement. However a distinction must be made between those inventions which terminated in a blind alley and those which led to the development of commercial machines.

Philadelphia, August 8, 1874.

F. S. Baldwin, Esq.

Dear Sir:

I have used for the last four months one of your large machines daily in this department and have no hesitation in saying it performs its work rapidly and reliably, and for the purpose used does the work of at least three men, with a certainty of correctness, and greater rapidity. For Railroad Companies, calculating mileages and tonnages, it is, I consider, invaluable.

Yours respectfully,

G. M. Taylor,
P.R.R. Co.

The writer has recently dismantled and reassembled one of these machines built fifty years ago and found with some slight adjustment necessary from wear that the machine functioned perfectly.

*The Odhner Machine.* Willgodt Theophile Odhner, a Swede, filed application on July 13, 1878, and was granted U. S. Patent No. 209416 on October 29, 1878, on a machine very similar in design to the Baldwin machine and embodying the two distinctive features of Mr. Baldwin's invention, the cam operated radial pins and the sliding wedge carrying mechanism. Odhner interchanged the Baldwin static and moving parts, placing the numeral wheels in the carriage and the selector mechanism in the frame as in the Thomas machine. Otherwise the disposition of parts and operation are very similar to those of the Baldwin machine. Odhner's other patents were dated as follows: Germany, November 19, 1878, No. 7393; Russia, applied for February 14, 1879, granted December 31, 1879 (O.S.), No. 2329 (No. 148 of 1879).

*Priority.* The earliest documentary [1] evidence for the Odhner machine available to the writer is the date of the application for the U. S. Patent, July 13, 1878. Baldwin's caveat shows a completed design on September 28, 1872.

*Independence.* External evidence. Is it possible that these two machines, externally similar in construction and operation and based on a combination of the same two essential mechanisms, should be invented independently? Consider the internal evidence. Preceding Odhner's claims in U. S. Patent 209416, occurs the following statement: "I do not claim, broadly, setting the teeth of the counting-wheels by means of an adjustable cam or cam-wheel, nor the use of a slide for causing the lateral movement of tenth-carrying teeth, nor the combination of a toothed counting and recording wheel." No other machine than Baldwin's had these essential features. It is of more than passing interest to note that this allowance of credit does not appear in foreign patents granted to Odhner.

It is not the purpose of this note to attempt an evaluation of Mr. Baldwin's first calculating machine, nor of his subsequent work, but rather to establish the validity of the statement that this machine is the direct and lineal prototype of all machines of that class which have erroneously been attributed to Odhner,

[1] Johannes Meyer, Pappenheim, Bavaria, has an Odhner illustration which has been in his possession since about 1904, and which bears the inscription, "Hand made model of the inventor in 1874." If this date be accepted at its face value, it definitely places the early Odhner model, leaving Mr. Baldwin with a margin of two years.

that Mr. Baldwin shall receive the credit which has been so long and so persistently withheld and which is his just due.

Thomas de Colmar was the inventor of the first practical non-reversible cycle calculator. Frank Stephen Baldwin was the inventor of the first practical reversible cycle calculator. Probably 90 per cent. of all calculators which have achieved commercial success since 1875 must be classed as direct successors of the Baldwin patent.

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## THE THEOREM ON THE MOMENT OF MOMENTUM.

By O. D. KELLOGG, Harvard University.

**1. Introduction.** In volume XXI of the MONTHLY, Professor Huntington[1] pointed out the inadequacy, and in some cases also the erroneousness, of the treatments of the theorem on the moment of momentum in the current elementary texts in mechanics. He then gave a correct and admirably simple treatment of the theorem in the case of uniplanar motion of a rigid body, on the assumption that a rigid body may be regarded as a system of a finite number of particles rigidly connected.

In conducting courses during the last four years, I found that not only were the elementary texts unsatisfactory with regard to this theorem, but that many of the classical books on mechanics were also. For this reason, and also because, in the interests of simplicity, Professor Huntington restricted himself in point of generality, I venture to offer the presentation which I have developed in my lectures. I do this under no misapprehensions as to the likelihood of novelty in method or results, but because the theorem is so important that it is high time that correct presentations of it became current.

**2. Statement of the Theorem.** We shall first consider a system of $n$ particles, which may be free, or subject to constraints. The hypotheses of the theorem are Newton's second law, and his third law extended as indicated below. These laws, of course, postulate the existence of absolute motion. They may be stated as follows:

II. *The product of the vector acceleration of a particle by its mass is proportional to the resultant of the forces acting on the particle.*

III. *The forces exerted by two particles on each other are equal in magnitude, and opposite in sense.* We assume further that they are *directed along the line joining the particles.*

Forces satisfying this extended third law we call *internal forces.* All others we call *external.*

By the *momentum* of a particle, we mean the product of its vector velocity by its mass. The velocity may be either the absolute velocity, or the velocity relative to some specified moving point. By the *moment* of a vector $v$, passing

[1] E. V. Huntington, "The Theorem of Rotation in Elementary Mechanics," this MONTHLY (*1914*, 315–320).

through a point $P$, with respect to a point $Q$, we mean the vector (or outer) product of $\boldsymbol{v}$ by the vector $\overrightarrow{QP}$, or $\overrightarrow{QP} \times \boldsymbol{v}$. By the *moment of momentum* of a particle relative to a point $Q$, we mean the moment with respect to $Q$ of the momentum of the particle relative to $Q$. The moment of a *system* of vectors relative to $Q$ will be understood to mean the vector sum of the moments with respect to $Q$ of the individual vectors.

With these, the usual definitions, and on the basis of the hypotheses II and III, we may now state the theorem:

THEOREM. *The time rate of change of the moment of momentum relative to a point $Q$ of a system of $n$ particles is equal to the moment relative to $Q$ of the external forces, if* and *only if* the point $Q$

(*a*) *has zero acceleration,*

(*b*) *is the center of mass of the system,* or

(*c*) *is such that its acceleration vector always passes through the center of mass.*[1]

**3. Proof.** The proof of the theorem is simple. Newton's second law may be written, with proper choice of units, in the form:

$$m_i \ddot{\boldsymbol{r}}_i = \sum_j \boldsymbol{F}_{ij} + \boldsymbol{F}_i, \qquad (i = 1, 2, \cdots n), \tag{1}$$

where $\boldsymbol{r}_i$ is the position vector of the $i$th particle drawn from a fixed origin, $O$, where $\boldsymbol{F}_{ij}$ is the internal force on the $i$th particle due to the $j$th, and where $\boldsymbol{F}_i$ is the resultant of the external forces on the $i$th particle. The extended third law states that $\boldsymbol{F}_{ji} = -\boldsymbol{F}_{ij}$, and that this vector is parallel to the vector joining the corresponding particles, or $\boldsymbol{r}_j - \boldsymbol{r}_i$. Hence $(\boldsymbol{r}_j - \boldsymbol{r}_i) \times \boldsymbol{F}_{ji} = 0$, or $\boldsymbol{r}_j \times \boldsymbol{F}_{ji} - \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = \boldsymbol{r}_j \times \boldsymbol{F}_{ji} + \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = 0$. If we now sum with respect to $i$ and $j$, we see that $\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = 0$, and this holds whether the vectors $\boldsymbol{r}_i$ are measured from a fixed point or not, provided they are measured from the *same* point.

We now introduce the general point of reference, $Q$. Let $\boldsymbol{q}$ denote the position vector of $Q$ measured from the fixed point $O$, and let $\boldsymbol{s}_i$ denote the position vector of the $i$th particle measured from $Q$, so that

$$\boldsymbol{r}_i = \boldsymbol{q} + \boldsymbol{s}_i. \tag{2}$$

We are now ready to form the equation between moments, by taking the vector products of the two members of each of the equations (1) by the corresponding $\boldsymbol{s}_i$ and adding. The result is

$$\sum_i \boldsymbol{s}_i \times m_i \ddot{\boldsymbol{r}}_i = \sum_i \boldsymbol{s}_i \times \boldsymbol{F}_i, \tag{3}$$

[1] The usual practice with respect to this theorem has been to prove it on the (usually tacit) assumption that $Q$ is fixed. It has then been applied in cases where $Q$ is not fixed. The more careful writers, of course, avoid this error, but I have not seen any statement equivalent to the above. Stäckel, in his article in the *Encyk. d. Math. Wiss.*, IV, 6 (1908), p. 475, makes no statement about the point $Q$. Routh states the theorem for the cases (*a*) and (*b*), and considers cases in which $Q$ may be a point of the instantaneous axis when the system is replaced by a rigid body; *Elem. Dyn. of a Rigid Body*, 5th ed. (1891), p. 179. Appell confines himself to cases (*a*) and (*b*); *Traité de Méc. Rat.*, 2d ed. (1904), vol. II, p. 23, p. 34. Webster states that the theorem holds for any point $Q$; *Dynamics* (1904), p. 96. Similarly, Jeans; *Theoret. Mech.* (1907), p. 296.

from which the moment of the internal forces is omitted, since, as we have seen above, it vanishes. The left-hand side of the equation may be transformed as follows:

$$\sum_i \boldsymbol{s}_i \times m_i \ddot{\boldsymbol{r}}_i = \sum_i \boldsymbol{s}_i \times m_i(\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{s}}_i) = \sum_i m_i \boldsymbol{s}_i \times \ddot{\boldsymbol{q}} + \sum_i \boldsymbol{s}_i \times m_i \ddot{\boldsymbol{s}}_i$$

$$= M\boldsymbol{s} \times \ddot{\boldsymbol{q}} + \frac{d}{dt} \sum \boldsymbol{s}_i \times m_i \dot{\boldsymbol{s}}_i,$$

where $\boldsymbol{s}$ is the position vector of the center of mass of the system relative to $Q$, and $M$ is the total mass. Thus, if $\boldsymbol{\sigma}_Q$ denote the moment of momentum of the system with respect to $Q$, and $\boldsymbol{M}_Q$ the moment of the external forces with respect to the same point, the left-hand side of the equation (3) becomes $M\boldsymbol{s} \times \ddot{\boldsymbol{q}} + \frac{d}{dt}\boldsymbol{\sigma}_Q$, and the right-hand side is $\boldsymbol{M}_Q$. The result may be written

$$\frac{d}{dt}\boldsymbol{\sigma}_Q = \boldsymbol{M}_Q + (-\boldsymbol{s}) \times M\ddot{\boldsymbol{q}}. \tag{4}$$

This is our goal. The result may be stated in alternative ways:

A. *The theorem on the moment of momentum becomes valid for any point $Q$, if we add to the moment of the external forces the moment with respect to the center of mass of the force which would give to the system, concentrated at $Q$, the motion of the point $Q$.* For the force in question is $M\ddot{\boldsymbol{q}}$, and the vector from the center of mass to $Q$ is $-\boldsymbol{s}$..

B. *The theorem on the moment of momentum is valid as stated in § 2.* For the conditions there imposed upon the point $Q$ are seen to be exactly those resulting from the vanishing of the additional moment in (4), *i.e.*, from the vector equation $\boldsymbol{s} \times \ddot{\boldsymbol{q}} = 0$.

**4. Remarks.** It will be noticed that no hypothesis of rigidity has been needed in the above developments. On the other hand, the results have been obtained only for a system of $n$ particles. To pass from such a system to continuous bodies, rigid or deformable, as is done in many texts, clearly involves new assumptions, for by no division of a continuous body do we ever arrive at a system of a finite number of particles. This lacuna may be met in a variety of ways; one may, for instance, postulate the legitimacy of the process of integration. Or, one may regard this process as heuristic, and postulate the results. Another alternative which seems to me to possess decided advantages of naturalness is to interpret Newton's second law in the form of the theorem usually called the momentum theorem: *the mass times the acceleration of the center of mass is proportional to the resultant of the external forces for any system, discrete or continuous.*[1]

[1] It is interesting to note that in the statement of his laws of motion Newton did not use the word *particle*, but *body*. The above interpretation is therefore natural. As stated, however, Newton's laws do not admit of immediate statement as equations, for the *change of motion* (which we now call mass times acceleration) of a *body* is not , without further convention, a single vector which can be proportional to the force vector. An easily accessible statement of Newton's laws in their original language is to be found in Webster, *Dynamics*, § 13, pp. 21, 22.

Under any circumstances, however, it will be noted that the theorems derived will hold with at least as great a degree of generality as any of the principles of mechanics which are based on Newton's laws and the assumption that the pairs of forces of reaction are directed along the lines joining their points of application.[1]

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## ON A TRANSFORMATION BY PAPER FOLDING.

By C. A. RUPP, University of Texas.

**1. Introduction.** He who delves in obscure corners of the mathematical world may well be pleased to find only two predecessors listed in his field in the standard histories and indexes. Cajori,[2] under the heading Paper Folding, confines himself to a quotation from Klein,[3] who devotes a paragraph to the work of Wiener and Row.

Klein states that in 1893 a German and a Hindu hit on the device of using paper folding in geometric constructions. Wiener[4] apparently found out how to construct the nets of the regular polyhedra by paper folding. His memoir has been inaccessible to the author, but its title would seem to indicate that there is no overlapping.

Beman and Smith were sufficiently interested in Klein's note concerning T. Sundara Rowe (or Row) to go to the trouble of hunting up the original book, which was published in India in 1893, and was therefore rather inaccessible to Occidental scholars. The book proved so interesting that they revised it and published it in this country, where it has had several editions.[5]

Several elementary analytic geometries amuse the freshmen with the following problem: Take a piece of paper having a straight edge, mark a point not on the edge, fold the paper so that a point of the edge coincides with the marked point, and prove that the creases so formed are tangent to a parabola whose directrix is the edge of the paper and whose focus is the fixed point.

The main object of this paper is to study and identify the mechanical folding transformation which turns a straight line into a parabola. Incidentally we shall show how to construct by paper folding the pedal and podoid of any given curve, as well as the negative pedal and negative podoid. We shall see that it is just as easy for the freshman to construct an ellipse or hyperbola by folding as it is to make the parabola by such a method. The central object of the

[1] Since the setting in type of this paper, Professor Murnaghan has kindly called to my attention the fact that Painlevé published a completely satisfactory statement and proof of the theorem with which we are concerned nearly thirty years ago, in his *Leçons sur l'intégration des équations différentielles de la mécanique* (Paris, A. Hermann, 1895, pp. 9–13). Painlevé states the theorem as in §2 above; it has not the same generality as in (A), §3, nor is his proof quite so elementary.

[2] Cajori, *History of Elementary Mathematics*, MacMillan, 1921, page 265.

[3] Klein, *Famous Problems in Elem. Geom.*, Beman and Smith, Ginn, 1897, p. 42.

[4] Dyck, *Katalog Munchener Math. Ausstellung von 1893*, Nachtrag, Seite 52.

[5] Row, *Geometric Exercises in Paper Folding*, third ed., rev. by Beman and Smith, Open Court Publishing Co., 1917.

paper is the consideration of folding as a mechanical means of effecting a transformation. Row seems not to have considered the possibility of transforming one curve into another by paper folding.

**2. Definitions.** Our outfit consists of a flat piece of paper on which are marked a fixed point $O$ and an arbitrary curve $C$. We will choose $O$ as the origin of our coördinate system, and we will call it the *pole* of the transformation.

*Operation 1* is defined as the process of forming the crease $c$ by folding the paper so that an arbitrary point $P$ falls upon the pole $O$. Note that to make this construction feasible we must have paper of reasonable transparency, as ordinary typewriter paper. This utilization of the transparency of the paper to guide us in making our folds accurately is a departure from the methods of Sundara Row. We also see that Operation 1 associates a line with every point in the finite plane with the exception of the pole, this line being the perpendicular bisector of the line joining the point and the pole.

*Operation 2* is the process of finding $M$, the mid-point of $OP$, as the intersection of the creases $c$ and $OP$.

*Operation 3* is the process of folding a crease $\gamma$ which will be tangent to a curve $C$. By a trial and error adjustment of the paper before we make the crease we can get the tangency about as closely as we wish. The accuracy is certainly comparable with that of drawing the tangent to a given curve with the aid of a ruler and a good eye.

*Operation 4* is defined as making a crease $\gamma'$ through the origin and perpendicular to $\gamma$. This is readily done by finding that crease through $O$ which makes the two halves of $\gamma$ coincide. Of course the problem of making a crease through a given point and at right angles to a given line is one of the simplest solved by Row.

If we call the intersection of $\gamma$ and $\gamma'$ the point $K$, we may define *Operation 5* as the process of determining $L$ as the image of the pole $O$ in $K$. Row would effect this quite simply by folding along $\gamma$ and, while the paper was still folded, pricking the paper through at $O$ with a sharp point. It seems a bit more sporting not to use a needle, but to proceed as follows: Make $\gamma$ and $\gamma'$. Refold the paper along the crease $\gamma$, and, without unfolding, crease the paper through $O$ perpendicular to $\gamma'$. The unfolded paper shows four creases, $\gamma$, $\gamma'$, and two creases parallel to $\gamma$, one passing through $O$ and the other, $\gamma''$, through the required point $L$, which is thereby determined as the intersections of $\gamma'$ and $\gamma''$.

These five constructions will be enough for this paper. The reader is urged to assure himself of the practicability of each of these folding operations.

**3. Theorems.** If the points of Operation 1 be taken at short intervals along the curve $C$, it will be seen that the corresponding creases $c$ envelop a curve $C'$, which we call the transform of $C$. In just a moment we shall prove that the creases $c$ do have an envelope, and we shall identify it, but for the present it seems convenient to consider the locus of $M$ as $P$ travels along the curve $C$.

THEOREM 1. *The transformation of a curve by Operation 2 repeated is a homothetic transformation, with the ratio of similitude one half.*

PROOF: By the definition of Operation 2, the point $M$ is half-way between the point $P$ and the origin. If $P$ moves along an arbitrary curve $C$, $M$ will generate a similar curve of half the size.

Let us now turn to the question of the envelope of the creases $c$. If we take the equation of the arbitrary curve $C$ in the form $y = f(x)$ referred to $O$ as the origin, and take the coördinates of the point $P$ on $C$ to be $(x, y)$, then, since the crease $c$ is the perpendicular bisector of $OP$, the equation of $c$, in the running coördinates $X, Y$, is

$$2(xX + yY) = x^2 + y^2.$$

The envelope of the creases $c$ is found by the ordinary methods of the calculus to be, in parametric form,

$$X = \frac{2xy + (y^2 - x^2)f'}{2(y - xf')}, \qquad Y = \frac{y^2 - x^2 - 2xyf'}{2(y - xf')},$$

where $f'$ denotes differentiation of $y = f(x)$.

THEOREM 2. *The transformation of a curve by Operation 1 repeated is equivalent to a homothetic transformation followed by a negative pedalization, the pole of both transformations being the origin.*

PROOF: By Operation 1 the crease $c$ is perpendicular to the segment $OP$ at its mid-point $M$. By definition, the first negative pedal of a curve is the envelope of the perpendiculars drawn to the radii vectores at their extremities. It follows that $C'$, the envelope of the creases $c$, is the first negative pedal of the $M$-curve, which, by Theorem 1, is homothetic to $C$.

We see that the folding transformation effected by the repetition of Operation 1 gives the same transform that we get by first shrinking $C$ about the origin to half its size, and then taking the negative pedal of the result. Let the student prove that first taking the negative pedal of $C$, and shrinking the result to half its size about the origin will again give $C'$.

THEOREM 3. *The inverse folding transformation is identical with the PODOIDAL transformation.*

PROOF: Reversing our machinery to see with what original curve $C$ we must begin if we wish our folding to yield us a given curve $C'$, we see that we are to pedalize $C'$ in the origin, and dilate to twice the size, *i.e.*, to take the reflection of the pedal of a curve in the tangents of that curve, a transformation which has been studied by Brocard,[1] Loria,[2] and others, under the name of the PODOID.

This identification of the folding transformation with a reasonably well-known birational point-line transformation shows us how to produce all kinds of curves. We needed only to have known that the podoid of a parabola in its

[1] Brocard, *Notes de bibliographie des courbes geometriques*, Bar le Duc, 1897, page 221. Brocard et LeMoyne, *Courbes Geometriques*, Vuibert, 1919 (Tome I), p. 27.

[2] Loria, *Ebene Kurven* (zweite auf. Teubner, 1910), II, Seite 324.

focus is its directrix to have easily foreseen the result of the problem in paper folding stated in the Introduction. Now that we have the clue to the transformation we can grind out theorems by the score. For example, under the folding transformation a unicursal quartic whose node is the pole will give a conic in general position.

Row,[1] in his chapter of conics, confines himself to the construction of points on the curves, with the exception of noting on page 116 that his crease is really tangent to the parabola. We wish to construct an ellipse as the envelope of its tangents. Remembering that the reflection of the focus in the tangents is the director circle, whose center is the other focus, and whose radius is the major axis, we see that $C$ is a circle surrounding the origin. More generally, we can state the

THEOREM 5. *The folding transform of a circle is a conic whose foci are the center of the circle and the pole of the transformation. If the pole is at the center of the circle, the transform is itself a circle. If the pole is within the circle, the transform is an ellipse. If the pole is outside the circle, the transform is a hyperbola whose asymptotes are perpendicular to the tangents to the circle from the pole of the transformation.*

The last part of this theorem is an excellent problem for a good class in analytic geometry.

Paper folding is a more powerful method of construction than one might think. Consider the problem of finding the pedal of an arbitrary curve $C$.

THEOREM 6. *The succession of Operations 3 and 4 upon the points of an arbitrary curve yields the points of the pedal in the origin.*

PROOF: The point $K$ on the pedal which corresponds to the point $P$ on the curve is found by dropping a perpendicular from $O$ on the tangent at $P$ of the curve $C$. But this point $K$ is the point of intersection of the creases $\gamma$ and $\gamma'$ of the Operations 3 and 4.

THEOREM 7. *The succession of Operations 3, 4, and 5 on the points of a curve C yields the points of the podoid with respect to the origin.*

PROOF: By the definition of the podoid, a point $P'$ on it is symmetric to the origin in the tangent to $C$ at $P$. The crease $\gamma'$ of Operation 4 is by hypothesis perpendicular to $\gamma$, which is tangent to $C$ at $P$, and hence the point $L$ of Operation 5, originally defined as the image of $O$ in the point $K$, is also the image of $O$ in the line $\gamma$. $L$ is accordingly the point $P'$ of the podoid.

**4. Note.** The process of paper folding that we have considered is essentially one involving the line of intersection of two planes. The author is now preparing a paper extending the idea of folding to space of three dimensions, and also to surfaces any part of which is applicable to any other part thereof.[2]

[1] Row, *loc. cit.*, Chap XIII, passim.

[2] For an analytical discussion of the parabola, ellipse, and hyperbola as obtained by paper folding see "Construction of the Conic Sections" by A. J. Lotka, *Scientific American Supplement* for Feb. 17, 1912, page 112. EDITORS.

# THE USE OF PROGNOSTIC TESTS IN SECTIONIZING COLLEGE FRESHMEN IN MATHEMATICS.[1]

By M. A. NORDGAARD, Antioch College.

With the advent of the large enrollments in our colleges and universities the question of effective freshman instruction has become paramount. Notably is this true in the departments of English and of mathematics. Latin and Greek are not continued by the student poorly prepared in these subjects; the natural sciences and modern languages have beginning courses in the college curriculum; the political sciences need not demand any specific preparation; but high school English and mathematics are the two branches that must be *continued* in college, either at the instance of the faculty or of the students themselves, and these departments are obliged to build on a substructure that is often both porous and miscellaneous.

As regards mathematics the most fruitful procedure has seemed to be to sectionize the freshmen according to ability, and let the subject matter as well as the mode of presentation be adapted to the status of the section. It is generally admitted that both native ability and specific preparation ought to enter into this classification. Ten years ago, at a certain eastern university, we sectionized the freshmen at the end of the first semester on the basis of their current work as appraised by the individual teacher, and formed as many sections for each hour as there were classes reciting at that hour. This seems still to be the current practice, though occasionally the classification is effected at the end of the first six or nine weeks. The plan has three inherent disadvantages: (1) there can be no standard as between the different sections of different hours, and in a small school the gradations are too few; (2) the classification comes too late; and (3) the personal equation generally prevents the appraisal of the individual instructors from being very satisfactory for this purpose.

When two years ago I took charge of the mathematics department at Antioch College, the dean and I decided to sectionize along rather finer lines and at the very beginning of the course. So on registration day every freshman took a mathematical placement examination consisting of two parts, each requiring one hour. For Part Two, I gave such questions from arithmetic, algebra, and geometry as are usually given in "finals" in high school. It failed utterly to differentiate the students into groups, and its results had to be disregarded: it was clearly *not the kind of test* to give students who had been on a vacation anywhere from three months to three years.

Part One proved to be a good "sieve," and we have used no other type of placement test since then. It also required one hour,—twenty minutes each on arithmetic, algebra, and plane geometry. It is a prognostic test, very much on the order of the prognostic tests used so successfully in grade and high schools the last ten years. In building it up I used as a pattern the recent psychological and vocational tests. It is a "hurdle test," and consists of thirty questions,

[1] Summary of a paper read at a meeting of the Ohio Section at Columbus, April 4, 1924.

whose answers can be quickly and easily graded. On the results we classified our two hundred and fifty freshmen (mathematics is required of all freshmen at Antioch) into five groups, *A*, *B*, *C*, *D*, *E*,—two divisions in each group. With a similar procedure our two hundred and eighty odd freshmen of the next year were divided into six groups.

It has proved a workable method of sectionizing. For, though the college time table permitted any freshman mathematics student to be shifted into any division without a conflict, we felt justified in shifting only about five per cent of the more than five hundred students sectionized in this manner during the semester, and an equal number at the end of the semester. For the others the placement test has sifted them into groups sufficiently homogeneous to work up an *esprit de corps*, to determine separate modes of presentation, and to advance at varying speeds.

But in order to investigate scientifically the relations of the scores on the prognostic test to records in current work, I made a few studies; I submit the results of two of these. As a gauge I used the scale propounded by A. R. Crathorne in his article "The Theory of Correlation Applied to School Grades" (see *The Reorganization of Mathematics in Secondary Education*, Chapter X). In his investigation Dr. Crathorne computed the correlation coefficients of subjects whose relationships were known to be close or remote. The following are indicative:

College algebra—trigonometry, same time, same instructor ..... 0.737,
College algebra—trigonometry, same time, different instructor ... 0.620,
Latin (boys), two consecutive courses ..... 0.680,
English (boys), two consecutive courses ..... 0.579,
Latin, two courses, two years intervening ..... 0.430,
English, two courses, two years intervening ..... 0.410,
Domestic science—civics ..... 0.260,
Manual training—Latin ..... 0.070.

From the study of seventy-four such correlation coefficients, Crathorne sets up the following scale:

| Extremely high | High | Medium | Low | Extremely low |
|---|---|---|---|---|
| 1.00 | .70 | .55 | .40 | .25 | 0 |

For my first study I selected from the 1922 freshmen a group of eighteen students in the top section, whose "external environment" had been identical, —tests at the same time and place, the same instructor, etc. The following correlations (Pearson formula) obtained:

Prognostic test—1st semester "finals" ..... 0.531,
Five week class test—semester "finals" ..... 0.422,
Prognostic test—1st semester marks ..... 0.294,
Five week class test—semester marks ..... 0.107,
Prognostic test—five week test ..... 0.211,
Prognostic test—Thurstone psychological test ..... 0.318.

Between college records and the scores of a prognostic test involving tools acquired in high school we should expect a correlation equal to that between two courses in English or Latin, with two years intervening,—that is, between .40 and .50.

The homogeneous character of the group may be a contributing reason for the low correlations. Even so we note that the placement scores bear a closer relation to the semester marks and "finals" than do the scores of the class test taken at the end of five weeks. As a gauge of a close relation I found the correlation between the marks of the first and second semesters to be 0.686.

I also made a study of two groups of this year's freshmen. The first group included fifteen students from sections *A* and *B*, taught by one instructor; the second group of twenty-six people were from sections *E* and *F* and taught by another instructor. The low correlation of the second group was traceable to four quite indifferent students whose vocational experience and inclinations gave them an unduly high grade in the arithmetic and geometry part of the placement examination. Their misplacement was soon detected and they were shifted. The algebra placement and the semester ranks had a high correlation. Following are the correlation coefficients:

| | Group *I* | Group *II* |
|---|---|---|
| Prognostic test—Semester ranks | 0.676 | 0.404 |
| Prognostic test—Average semester tests | 0.664 | 0.447 |
| Algebra placement—Average semester tests | 0.523 | 0.760 |
| Thurstone psychological test—Average semester tests | 0.365 | 0.239 |
| Thurstone psychological test—Semester ranks | 0.220 | 0.316. |

In secondary education, as observed in the University High Schools of Iowa and of Minnesota and the experimental schools of Teachers College, New York City, one of the chief values of prognostic tests has been to gauge what results may be expected by the teacher in charge of a group. And we have found the prognostic tests at Antioch of no less value in college work. Knowing the calibre of inferior and medium students we need not expect the impossible either from them or from the instructor. The test also helps to single out students from whom much may be expected, at the very beginning of their course, before they get into slovenly habits of study due to lack of competition.

# QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

**The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.**

## DISCUSSIONS.

### I. A NUMBER THEORETIC FUNCTION SATISFYING THE FUNCTIONAL EQUATION $\psi(m \cdot n) = \psi(m) + \psi(n)$.

BY A. J. KEMPNER, University of Illinois.

1. *Definition of a number theoretic function satisfying the functional equation:*

$$\psi(m \cdot n) = \psi(m) + \psi(n).$$

We assume $p_1, p_2, \cdots$ to be distinct primes and define $\psi(m)$ in the following manner:

1. For $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, $\alpha_1, \alpha_2, \cdots, \alpha_n$ integers $\geqq 1$, let $\psi(m) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$;

2. $\psi(1) = 0$.

Then we shall have for positive integral values of $m, n, k$, $\psi(mn) = \psi(m) + \psi(n)$, $\psi(m^k) = k \cdot \psi(m)$, so that $\psi(m)$ satisfies the functional equation of the logarithmic function.

2. *Extension of $\psi(m)$ to include rational values of $m$.*

Let $p_1, \cdots, p_n, q_1, \cdots, q_\nu$ be distinct primes, $m_1 = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, $m_2 = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_\nu^{\beta_\nu}$, the $\alpha$'s and $\beta$'s positive integers, $m = m_1/m_2$. We set $\psi(m) = \psi(m_1/m_2) = \Sigma\alpha - \Sigma\beta$.

3. *Further extension of $\psi(m)$ to include argument values $m = (m_1/m_2)^{\alpha/\beta}$.*

We first observe that every number of the form $(m_1/m_2)^{\alpha/\beta}$ can be written in the form [1]

$$\frac{p_1^{\alpha_1/\alpha_1'} \cdots p_n^{\alpha_n/\alpha_n'}}{q_1^{\beta_1/\beta_1'} \cdots q_\nu^{\beta_\nu/\beta_\nu'}}, \tag{I}$$

where the $p$'s and $q$'s are defined as above, all $\alpha$'s and $\beta$'s positive integers and each pair $\alpha_i, \alpha_i'$ and each pair $\beta_j, \beta_j'$ relatively prime. We admit also that the numerator or denominator, or both, may be unity.

We shall need the fact that the representation of a number in the form I is unique. For simplicity of notation we indicate the simple proof of a special case. Assume

$$\frac{p_1^{\alpha_1/\alpha_1'} \cdot p_2^{\alpha_2/\alpha_2'}}{q_1^{\beta_1/\beta_1'} \cdot q_2^{\beta_2/\beta_2'}} = \frac{r_1^{\gamma_1/\gamma_1'} \cdot r_2^{\gamma_2/\gamma_2'}}{s_1^{\delta_1/\delta_1'} \cdot s_2^{\delta_2/\delta_2'}}$$

[1] It is understood that whenever a rational exponent occurs, the real positive value of the radical is to be chosen.

to be two distinct representations I of the same number. Then

$$\frac{(p_1^{\alpha_1\alpha_2'} \cdot p_2^{\alpha_2\alpha_1'})^{1/\alpha_1'\alpha_2'}}{(q_1^{\beta_1\beta_2'} \cdot q_2^{\beta_2\beta_1'})^{1/\beta_1'\beta_2'}} = \frac{(r_1^{\gamma_1\gamma_2'} \cdot r_2^{\gamma_2\gamma_1'})^{1/\gamma_1'\gamma_2'}}{(s_1^{\delta_1\delta_2'} \cdot s_2^{\delta_2\delta_1'})^{1/\delta_1'\delta_2'}}.$$

Multiply across and raise both sides to the power $(\alpha_1' \cdot \alpha_2' \cdot \beta_1' \cdot \beta_2' \cdot \gamma_1' \cdot \gamma_2' \cdot \delta_1' \cdot \delta_2')$; comparing both sides and remembering the uniqueness of factorization of integers into prime factors, the uniqueness of the representation follows immediately. We admit now as argument values all numbers $(m_1/m_2)^{\alpha/\beta}$ and define, thus including our previous definitions:

*For a given* $m = (m_1/m_2)^{\alpha/\beta}$, *write* $m$ *in the form* I *and let*

$$\psi(m) = \Sigma\alpha_i/\alpha_i' - \Sigma\beta_j/\beta_j'.$$

*If* $m_1 = 1$, *we take of course* $\Sigma(\alpha_i/\alpha_i') = 0$, *if* $m_2 = 1$, $\Sigma(\beta_j/\beta_j') = 0$. The functional equation $\psi(mn) = \psi(m) + \psi(n)$ still holds on account of the uniqueness of (1).

**4.** *Proof that* $\psi(m)$ *covers the half plane everywhere dense.*

We consider the graph of $y = \psi(x)$ in a rectangular system of coördinates. $\psi(x)$ is defined for an everywhere dense set of real argument values. We show that the function as defined in § 3 is represented by a set of points which cover everywhere dense the whole half plane $x > 0$. Everywhere dense means that if we select any point in this half plane and draw around it a circle of arbitrarily small radius, there will always be at least one point of the graph inside the circle,—and therefore an infinite number of points.

In a paper which I have not seen since twelve years and which is not available to me at present, "Axiomatische Untersuchungen über die Vektoraddition," 1909, R. Schimmack proved, I believe, that every solution of the functional equation $\psi(m \cdot n) = \psi(m) + \psi(n)$ which is discontinuous must be very strongly discontinuous, namely, of the type described above.

We make use of the following classical theorem on prime numbers:[1] *For a given arbitrarily small positive* $\eta$ *there exists an integer* $N = N(\eta)$ *such that for every* $n > N$ *there lies at least one prime between* $n$ *and* $n(1 + \eta)$.

All the more, there must be a prime between $n(1 - \eta)$ and $n(1 + \eta)$. To prove the theorem of this paragraph it is sufficient to show the following: *Given any* $\gamma > 0$ *and any rational* $\alpha \neq 0$, *then we can find three primes* $p_1$, $p_2$, $p_3$ *such that* $|(p_1 \cdot p_2/p_3)^\alpha - \gamma| < \epsilon$, $\epsilon$ *arbitrarily small* $> 0$.

*Proof:* Let $\gamma^{1/\alpha} = c$, $(\gamma + \epsilon)^{1/\alpha} = c + \epsilon_1$, $(\gamma - \epsilon)^{1/\alpha} = c - \epsilon_2$, and $\epsilon_3$ the smaller of $\epsilon_1$, $\epsilon_2$, so that $\epsilon_3 \to 0$ together with $\epsilon$. We have to show

$$-\epsilon < \left(\frac{p_1p_2}{p_3}\right)^\alpha - \gamma < \epsilon,$$

$$(\gamma - \epsilon)^{1/\alpha} < \frac{p_1p_2}{p_3} < (\gamma + \epsilon)^{1/\alpha},$$

$$c - \epsilon_3 < \frac{p_1p_2}{p_3} < c + \epsilon_3.$$

[1] See, for example, Landau, *Handbuch der Primzahlen*. The theorem is fundamental.

Choose $\epsilon_4 = \epsilon_3/c$, then also $\epsilon_4 \to 0$ with $\epsilon$, and

$$\frac{p_1p_2}{c} \cdot \frac{1}{1-\epsilon_4} > p_3 > \frac{p_1p_2}{c} \cdot \frac{1}{1+\epsilon_4}.$$

Finally introduce $\epsilon_5 > 0$ so that simultaneously

$$\frac{1}{1-\epsilon_4} > 1+\epsilon_5, \qquad \frac{1}{1+\epsilon_4} < 1-\epsilon_5;$$

obviously $\epsilon_5 \to 0$ with $\epsilon$. Then our preceding inequality is satisfied if we find three primes satisfying

$$\frac{p_1p_2}{C}(1-\epsilon_5) < p_3 < \frac{p_1p_2}{C}(1+\epsilon_5).$$

Let $N$ be a number so large that for $n > N$ there lies a prime between $n(1-\epsilon_5)$ and $n(1+\epsilon_5)$. Then select for $p_1$ and $p_2$ two primes so large that $p_1p_2/c > N$.

**5.** *Extension of $\psi(m)$ to include all positive argument values.* Our last step shall be to extend our function again, this time in such a manner that it shall be defined for all positive values of the argument, and so that the functional equation shall still hold. The method for doing this is well established, provided we accept the theorem of Zermelo that every set may be well ordered. Indeed, all that we need to do is to paraphrase a few paragraphs from a paper by Hamel [1] in the sense that we everywhere apply multiplication where he employs addition.

The argument values for which $\psi(x)$ is defined up to the present consist exactly of all numbers of the form $a^\alpha$, $a$ any positive rational number, $\alpha$ any rational number. This set we exclude from the continuum of positive numbers. Let the remaining set be $C'$, which is, since the deleted set is denumerable, still of the density of the continuum. Assume the set well ordered and $a_1$ its first element ($a_1$ some number not of the form $a^\alpha$). We then consider the denumerable set $a_1^{\alpha_1}$, $\alpha_1$ ranging over all rational numbers $\neq 0$, and define $\psi(a_1^{\alpha_1}) = \alpha_1$. For an $x$ of the form $a^\alpha \cdot a_1^{\alpha_1}$ we define $\psi(x) = \psi(a^\alpha) + \psi(a_1^{\alpha_1})$. Since no $a^\alpha$ equals an $a_1^{\alpha_1}$ and no product of an $a^\alpha$ and an $a_1^{\alpha_1}$ equals an $a^\alpha$, there can be no interference between the old functional values and the newly introduced ones. The new function is in a sense superimposed upon the old one, its argument values fit into lacunæ left by the arguments of the old function.

We continue in the same manner. We now imagine from the positive set $C'$ the (denumerable) subset $a^\alpha \cdot a_1^{\alpha_1}$ deleted. The remaining set, still of the density of the continuum, we assume to have a first element, $a_2$, by virtue of Zermelo's theorem. Consider the set of numbers $a_2^{\alpha_2}$, $\alpha_2$ rational, $\neq 0$. Then no $a_2^{\alpha_2}$ = any $a^\alpha \cdot a_1^{\alpha_1}$, or, which is the same, no $a^\alpha \cdot a_1^{\alpha_1} \cdot a_2^{\alpha_2} = 1$. Defining $\psi(a_2) = \alpha_2$ and agreeing that $\psi(a^\alpha \cdot a_1^{\alpha_1} \cdot a_2^{\alpha_2}) = \psi(a^\alpha \cdot a_1^{\alpha_1}) + \psi(a_2^{\alpha_2})$, the functional equation will still hold.

Proceeding thus a finite number of times we at each step enlarge the domain of our argument values, so that the argument may assume any number of the

[1] *Mathematische Annalen*, vol. 60 (1905), pp. 459–462.

form $a^{\alpha}\cdot a_1^{\alpha_1}\cdot a_2^{\alpha_2}\cdots a_k^{\alpha_k}$, where $a$, $a_1$, $a_2$ $\cdots$, $a_k$ have the property that they are independent in the sense that no $a_\nu^{\alpha_\nu} = a_1^{\alpha_1}\cdot a_2^{\alpha_2}\cdots a_{\nu-1}^{\alpha_{\nu-1}}$.

The question whether one can in this manner, by consecutive selection of an infinite number of $a$'s, obtain for the argument all positive values, is the same question as to ask whether it is possible to find a multiplicative, or logarithmic, base of all positive numbers, that is, whether it is possible to find an infinite (denumerable) set of numbers $a$, $a_1$, $a_2$ $\cdots$ so that every positive number is represented in the form $a^{\alpha}\cdot a_1^{\alpha_1}\cdot a_2^{\alpha_2}\cdots$ with rational exponents and the representation unique. Following nearly literally Hamel, we argue as follows, always admitting Zermelo's theorem: We can decide of any positive number $x$ whether or not it belongs to the base by making the following agreement: Let $X$ be the "Abschnitt" segment belonging to the element $x$ of the original set. Either $x$ is representable in the form $a^{\alpha}\cdot a_1^{\alpha_1}\cdot a_2^{\alpha_2}\cdots$ by a finite number of elements $a_i$ belonging to $X$, or it is not so representable. In the latter case we count $x$ to our base, otherwise not. With this agreement there can be no relation of the form $a^{\alpha}\cdot a_1^{\alpha_1}\cdot a_2^{\alpha_2}\cdots a_\nu^{\alpha_\nu} = 1$ with positive rational exponents between any finite number of base elements $a$, $a_1$, $a_2$ $\cdots$, for if there existed such a relation, one of these elements, for example $a_2$, would come last in the original arrangement, that is, $a_2$ is preceded in the original arrangement of the continuum by $a$, $a_1$, $a_3$, $\cdots$, $a_\nu$. This element ($a_2$) could be expressed in the form $a^{\alpha}\cdot a_1^{\alpha_1}\cdots$ by a finite number of elements $a$, $a_1$, $a_3$, $\cdots$ preceding $a_2$, thus contradicting our hypothesis.

From the method of construction of the base follows immediately that the representation, when possible at all, is unique.

Assume now that $x$ is no base element, then $x$ is always representable in the form $a^{\alpha}\cdot a_1^{\alpha_1}\cdots$, where $a$, $a_1$, $a_2$ $\cdots$ are elements of the base. For, if there were numbers which are neither base elements nor representable in this form, then there would exist among these numbers, since they form a subset of a well-known ordered set, a first one, say $y$. By assumption $y$ could then be represented in the form $a^{\alpha}\cdot a_1^{\alpha_1}\cdots$ by a finite number of preceding elements. But these preceding elements must either be base elements $a_i$ or they can themselves be represented by base elements, and since there are only a finite number of elements involved, the same conclusion holds for $y$.

Readers familiar with Hamel's paper will recognize that § 5 may also be described as follows: By passing over to logarithms from our numbers $a^{\alpha}\cdot a_1^{\alpha_1}\cdot a_2^{\alpha_2}$ $\cdots$, we obtain the elements of Hamel's base, $\alpha c + \alpha_1 c_1 + \alpha_2 c_2 + \cdots$, where $c_i = \log \alpha_i$, $c = \log \alpha$.

We have tried in this note to push back, as far as is easily feasible, the application of Zermelo's theorem, since from the point on where this theorem is invoked, the construction of the function is based on the assumed, but not really effected, "Wohlordnung" of the continuum.

## II. On the Number of Divisions in Finding a G.C.D.

By H. Grossman, Student, College of the City of New York.

*The number of divisions to be effected in finding the G.C.D. of two numbers by the Euclidian algorithm does not exceed five times the number of digits in the smaller number (when this number is written in the scale of ten).*[1]

Proof: Let $a_n$ and $a_{n+1}$ be the two numbers $(a_n < a_{n+1})$.

Finding their G.C.D. by the Euclidian algorithm:

$$\begin{aligned}
a_{n+1} &= a_n m_n + a_{n-1} \qquad (0 < a_{n-1} < a_n),\\
a_n &= a_{n-1} m_{n-1} + a_{n-2} \qquad (0 < a_{n-2} < a_{n-1}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
a_3 &= a_2 m_2 + a_1 \qquad (0 < a_1 < a_2),\\
a_2 &= a_1 m_1,
\end{aligned}$$

where $a_1$ is the G.C.D. of $a_n$ and $a_{n+1}$, and it requires $n$ divisions to find it.

We wish to prove that $n$ does not exceed five times the number of digits in $a_n$. Obviously $a_n$ is a minimum for a given $n$ when $m_{n-1} = m_{n-2} = \cdots = m_2 = a_1 = 1$ and $m_1 = 2$. (If $m_1 = 1$, $a_2 = a_1$.) In this case, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$, etc. That is, the smallest pair of numbers to find whose G.C.D. requires $n$ divisions are the $n$th and $(n+1)$th numbers in the sequence 1, 2, 3, 5, 8, 13, $\cdots$, in which $a_n = a_{n-1} + a_{n-3}$.

First we will prove that $a_{i+5} > 10a_i$, $(i = 1, 2, \cdots)$. This is seen to be true when $i = 1, 2, 3$. When $i > 3$, we note that $a_i = a_{i-1} + a_{i-2} = 2a_{i-2} + a_{i-3}$; whereas $a_{i+5} = 2a_{i+3} + a_{i+2} = 3a_{i+2} + 2a_{i+1} = \cdots = 21a_{i-2} + 13a_{i-3}$. Hence $a_{i+5} > 10a_i$.

From this it follows that, if $n > 5k$, $a_n$ has at least $k+1$ digits; so that $n$ does not exceed five times the number of digits in $a_n$.

## III. On the Exponential Notation for Circular Functions.

By H. L. Slobin, University of New Hampshire.

From De Moivre's Theorem,

$$(\cos\theta + i\sin\theta)^{p/q} = \cos(p\theta/q) + i\sin(p\theta/q);$$

and, when $\theta = \pi$,

$$2\cos(p\pi/q) = (-1)^{p/q} + (-1)^{-p/q}, \qquad 2i\sin(p\pi/q) = (-1)^{p/q} - (-1)^{-p/q},$$

where $(-1)^{p/q}$ means the $p$th power of the complex $q$th root of $-1$ having the least positive amplitude.

If we let $(-1)^{1/q} = x$, we have

$$(1)\quad 2\cos(p\pi/q) = x^p + x^{-p}, \qquad (2)\quad 2i\sin(p\pi/q) = x^p - x^{-p}.$$

[1] This theorem was proved by G. Lamé, *Comptes Rendus*, Paris (1844), vol. 19, pp. 867–9. For a number of references to this and similar results, see L. E. Dickson, *History of the Theory of Numbers*, vol. 1, p. 394. Editor.

From these two forms we have of course similar forms for all the circular functions.

The purpose of this note is to call special attention to the usefulness of these forms as a direct and elementary instrument for the solution of problems which otherwise necessitate a knowledge of advanced analysis.

The following problem from the *Educational Times* will illustrate a particular use of these forms.

*Problem:* If $\Sigma \cos^4(\pi/7) = \cos^4(\pi/7) + \cos^4(2\pi/7) + \cos^4(3\pi/7)$, find $\Sigma \cos^4(\pi/7)$, $\Sigma \sin^4(\pi/7)$, $\Sigma \tan^4(\pi/7)$, $\Sigma \sec^4(\pi/7)$.

Solution: By (1), $16\cos^4(\pi/7) = (x + x^{-1})^4$, $16\cos^4(2\pi/7) = (x^2 + x^{-2})^4$, $16\cos^4(3\pi/7) = (x^3 + x^{-3})^4$. Hence, since $x^7 = -1$, if we let $x^6 - x^5 + x^4 - x^3 + x^2 - x = k$, we have $\Sigma \cos^4(\pi/7) = (18 + 5k)/16$.

But from $x^7 + 1 = (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) = 0$, we have $k = -1$, so that $\Sigma \cos^4(\pi/7) = 13/16$.

Similarly,

$$\Sigma \sin^4(\pi/7) = (18 - 3k)/16 = 21/16,$$
$$\Sigma \tan^4(\pi/7) = 976k + 1347 = 371,$$
$$\Sigma \sec^4(\pi/7) = 16(31k + 57) = 416.$$

Obviously these formulæ may be used to verify the known trigonometric identities; and they also often lead to interesting forms not so readily obtained otherwise.

Thus let $p/q = r$. Then $2\cos r\pi = (-1)^r + (-1)^{-r}$, and we have immediately $4\cos^2 r\pi = (-1)^{2r} + (-1)^{-2r} + 2 = 2\cos 2r\pi + 2$, or on proceeding to the limit, since the cosine is a continuous function, and $r\pi$ may be made to approach any arbitrary angle $\alpha$, $2\cos^2\alpha = 1 + \cos 2\alpha$.

Or again, $2\cos nr\pi = (-1)^{nr} + (-1)^{-nr}$, which, if $n$ is odd, is equal to

$$[(-1)^r + (-1)^{-r}][(-1)^{(n-1)r} - (-1)^{(n-3)r} + \cdots + (-1)^{-(n-1)r}],$$

so that $(\cos n\alpha)/\cos\alpha = 2\cos(n-1)\alpha - 2\cos(n-3)\alpha + \cdots \pm 1$. Thus

$$\cos 9\alpha/\cos\alpha = 2\cos 8\alpha - 2\cos 6\alpha + 2\cos 4\alpha - 2\cos 2\alpha + 1.$$

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## RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

### REVIEWS.

*The Mathematical Theory of Relativity.* By A. S. EDDINGTON. Cambridge University Press, 1923. ix + 247 pages.

This book, as stated in the preface, is the outgrowth of a mathematical supplement to the French edition of the author's "*Space, Time and Gravitation.*" While, in its present form, its direct dependence on the earlier work has been minimized, it does presuppose such of the less technical knowledge of the theory

of relativity as is there given. Assuming these general ideas, the author attempts to provide a complete and consistent mathematical treatment of the subject. Mathematical in structure only, for the point of view is physical throughout. As the discussion by no means follows the beaten track, we proceed to give a somewhat detailed account of the contents.

The starting point is the four-dimensional space-time manifold. Its geometry is taken as Riemannian, *i.e.*, governed by a geometrical invariant $ds^2$, expressed as a quadratic differential form in the differentials of the coördinates. Physically, this is the interval, on which all observation is based. By noting that in a "small" region the coefficients may be taken as constant, and that in any case this is the simplest possibility, we are led to the flat space-time of special relativity. On physical grounds it is found that

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2,$$

essentially, where $c$ is the velocity of light. The derivation of the Lorentz transformation, and of the usual formulæ and consequences of the special theory then follow.

The chief method used in passing from the results of the special to those of the general theory is the Principle of Equivalence, which asserts that, for limited regions, the laws governing phenomena in the general theory are identical with those of the restricted theory. While this principle is frequently used, we are warned that it is rather a suggestion than an infallible dogma, and must be constantly checked by experiment. This warning is at variance with the reverence for the principle of equivalence expressed by most early writers on relativity including Eddington himself.

To treat the more general space-time, the tensor calculus is introduced. The physical meaning of tensors, as well as the necessity of their use in studying physical questions, is made clear. The difficulties apt to be experienced with the manipulations of tensors will be lightened owing to the full explanations given.

Gravitation is handled by assuming that in empty space the coefficients of the quadratic form

$$ds^2 = \Sigma g_{ij}dx_idx_j$$

satisfy the equations:

$$G_{ij} = \lambda g_{ij},$$

where $G_{ij}$ is the contracted Riemann-Christoffel curvature tensor ($B_{ijk}{}^k$), and $\lambda$ is either zero or a small constant. The Schwarzschild solution of these equations for the "symmetrical, static" field is derived by direct computation, given in quite clear and elegant form. Then, assuming further that the paths of particles are geodesics, and those of light rays geodesics of zero length, the three well-known experimental discrepancies with the Newtonian theory are discussed. Effects on the moon's motion are calculated by approximate methods, and found to be of negligible amount.

In the relativistic description of mechanics and hydrodynamics, the material energy tensor $T_{ij}$ makes its appearance. We note that the author defines the proper-density of a perfect fluid as the scalar obtained by contracting this tensor, which is not the usual convention. The analogy of the vanishing of the divergence of the energy tensor with the classical equations of hydrodynamics is pointed out. Further, by identifying this tensor with one involving the $g_{ij}$ whose divergence vanishes identically, we are led to the equations of gravitation in the presence of matter, the generalized Poisson equation:

$$G_j^i - \tfrac{1}{2}g_j^i(G - 2\lambda) = -8T_j^i.$$

The relation of this law to the Principle of Stationary Action is considered, though the author does not consider this Principle so fundamental as some writers do.

The intuitive conception of curvature is explained at some length, and a geometrical statement of Einstein's law of gravitation is attempted. The spherical world of de Sitter, and the cylindrical world of Einstein are described, and their relative merits compared.

Electricity, and its relation to the new theory, is then taken up. The general tensor form of Maxwell's equations is obtained by the principle of equivalence. Various consequences of them are drawn, including a justification of the assumption that light rays pursue nul-geodesics.

The concluding chapter of the book is devoted to the theories which attempt to correlate gravitational and electromagnetic phenomena by using geometries more general than that of Riemann. The author first gives Weyl's theory, described as "the greatest advance in the relativity theory after Einstein's work," and then a more general geometry of his own. In this geometry, the starting point is the $\Gamma_{jk}^i$ which occur in the equations of parallel displacement. As these also occur in the equations of the geodesics, this geometry is identical with the "geometry of paths" of Veblen and Eisenhart.[1] A tensor analogous to the ordinary curvature tensor is set up, and from it and its derived tensors, expressions are constructed which may be identified with physical quantities on the basis of similarity of properties.

A noteworthy point here is Eddington's distinction between true or natural geometry, like the Riemannian geometry which "actually" holds for space-time; and "world geometry" which is merely a geometry providing graphical representation for physical relations like the $pv$ space of an indicator diagram. He considers Weyl's geometry and his own generalization as mere world geometries which, however, are useful for the additional light they throw on the actual physical relations. This view is quite consoling to the pragmatist who considers the substitution of the ten $g_{ij}$ of Einstein for the single potential of Newton, forced on him by experimental evidence, quite sufficient complication without replacing these by forty $\Gamma_{jk}^i$ to achieve a formal symmetry of gravitation and electricity.

[1] *Cf.* L. P. Eisenhart, *Annals of Mathematics*, vol. 24 (1923), p. 367.

Throughout, the book is quite clearly written (a quality too rare in texts on relativity) and is to be heartily recommended to any one seeking a comprehensive grasp of the theory.

PHILIP FRANKLIN.

*Introduction to the Mathematical Theory of the Conduction of Heat in Solids.* By H. S. CARSLAW. Second edition, completely revised. London, Macmillan and Co., Ltd., 1921. xii + 268 pages. Price 30 shillings.

With the possible exception of the theory of vibrating strings, there is scarcely any chapter in physics which has given a stronger impetus to the development of mathematical analysis than the theory of the conduction of heat. Both theories were originally expanded in a purely formal manner; rigorous justifications came later.

The typical non-stationary problem in the conduction of heat leads to the following mathematical considerations. There is given a set of functions depending upon one or several space variables $x$, $y$, $\cdots$ and a time variable $t$, which functions satisfy a partial differential equation, the equation of conduction. It is required to pick out a sub-set of solutions of the differential equation which satisfy the boundary conditions for all values of $t$. The resulting functions ordinarily satisfy a condition of orthogonality with respect to the space variables on a certain range. The next step is to express the solution of the given problem in terms of the functions in the sub-set in such a manner that the solution approaches the prescribed initial state as a limit when $t$ approaches zero.

There are quite a few delicate points involved in this process. For instance, are all the roots of the characteristic equation which determines the particular solutions real? Or, a much deeper question: Can the initial distribution of the heat in the solid be expressed in terms of the reduced set obtained by letting $t$ approach zero in the expression for the particular solutions determined by the boundary conditions? Finally, in what sense does the solution approach the initial state when $t$ approaches zero?

These questions are fundamental for the mathematical theory of the conduction of the heat, and it seems to the reviewer that a book on this subject has to be judged largely upon the kind of treatment it accords to these considerations. It is natural that older treatises on the theory in question should be lacking in this respect, but it is surprising how much inexactness is still to be found in much-used texts of fairly recent date.

Professor Carslaw's *Introduction to the theory of Fourier's series and integrals and the mathematical theory of the conduction of heat* appeared in a second edition some years ago. The first edition was perhaps the best comprehensive treatise on the subject, though, maybe, it did not fulfill all requirements as to rigor and disposition of the subject matter even at the time of its publication. In the revision the few sins against rigor seem to have been pretty well amended and the book stands the test proposed above with ease. But the general outlook of the book is not modern. It is true that there are a good many details that are

treated according to modern ideas; however, these details are added or interpolated in the old text which as a whole is not conceived in a modern spirit. Further, the book pays very little attention to the questions concerning the equation of conduction which have come to the foreground since the first edition appeared. Still this is pardonable in a book which is called an Introduction.

In the second edition the two parts of the book have been published in separate volumes of which the second one deals with the conduction of heat. This review is concerned with the second volume exclusively. In the revision, the second part has been modestly increased, namely from 244 to 268 pages, and the whole book has been rewritten. In the following we shall point out the essential differences between the two editions.[1]

The changes in the first six chapters are mostly of an editorial nature. These chapters deal with the equation of conduction, Fourier's ring, linear flow of heat in various solids, two-dimensional problems and the flow of heat in a rectangular parallelepiped. We notice a more rigorous treatment of questions of convergence of Fourier's series and integrals. The classical controversy between physicists and geologists concerning the age of the earth is now reduced to its right proportions since radioactivity has to be taken into account.

Chapter 7 deals with the conduction of heat in cylinders. A few new problems have been worked out; the theory of Bessel's functions has been elaborated more in detail and ample references are given to the literature. Additional material on Bessel's functions is to be found in an appendix at the end of the book. In chapter 8, the greater part of which is devoted to the flow of heat in the sphere, we notice a new problem involving a cone.

The theory of sources and sinks is presented in chapter 9. Here the classification of the different types of sources has been systematized. Some additions have been made to the method of images but the greater part of the subject matter under this heading in the first edition has been transferred to the following two chapters. These chapters deal with the use of Green's function and of contour integration respectively. The latter chapter is new and so is the greater part of its material. The distribution of the subject matter between the two chapters is rather arbitrary; as a matter of fact, almost every problem in chapter 9 is solved by means of contour integration and could just as well have been placed in chapter 10 instead. In the opinion of the reviewer these two chapters are the most interesting and instructive ones in the whole book. A considerable part of the material is based upon the author's own investigations.

In the last chapter we come to integral equations. This chapter is altogether new but deplorably short. In the third edition we should like to see this part of the theory placed, if not at the beginning, at any rate not at the end of the book. The advantages which could be derived from such an arrangement are essentially of a systematic nature. This would provide a fixed frame of reference to which the special developments could be attached. As it is now, the abun-

[1] The first edition does not seem to have been reviewed in the MONTHLY. However, the book has been known so many years in wide circles that it is not worth our while to give a detailed description of it.

dance of special methods and devices tends to hide the underlying general ideas in a way which is not desirable from the didactic point of view.

We have still another pious wish. The mathematical theory of heat conduction is merely a part of the theory of the equation of conduction, a partial differential equation of the parabolic type. Why not devote a chapter to the general theory of this equation? The quasi-analytic character of the general solution considered as a function of $t$ should be brought out. There is plenty of simple material in the investigations of Holmgren and Levi, for instance, which would interest the mathematically inclined reader. We should regard a reference to Goursat's presentation of these modern investigations as the least the author could do.

We have frankly pointed out some of the shortcomings of the book. They are not many and after all is said and done an excellent book remains. The reader who has time and a certain amount of perseverance at his disposal, enough to read the book carefully, will absorb not only the theory of the conduction of heat but also a fair portion of the classical methods of mathematical physics. He who wants to use the book for reference purposes will find the text clear and easy to read and will appreciate the carefully compiled bibliography at the end.

EINAR HILLE.

*Charts and Graphs.* By K. G. KARSTEN, consulting statistician, with an introduction by CARL SNYDER, chief statistician of the Federal Reserve Bank of New York. New York, Prentice-Hall, Inc., 1923. xl + 724 pages. $4.00.

The first part of the book carries forward the pioneer work of Brinton,[1] and presents in an interesting and attractive way recent developments in popular methods of representing facts. To business executives who are unable to appreciate mathematical analysis, the appeal must be to the eye; they must *see* the facts. The necessity for such an appeal is recognized and is met by the illustration and description of an almost endless number of charts, graphs and diagrams. Even one who has made an effort to keep in contact with progress in this phase of statistical development will be surprised at the wealth of material available. Evidently the creation of a new kind of chart has made a strong appeal to the ingenuity and imagination of the statistician who depends for his interpretation of facts upon a set of drawing instruments. The author deserves credit not only for his production of a work of encyclopedic nature but for his choice of the order of presentation which is planned to emphasize the evolution of the more complicated charts from a few fundamental ones. The pleasure of following a chart through its various stages of development will be especially appreciated by the critical reader.

The second part of the book ventures far enough into the field of mathematics to discuss logarithmic charts, the laws of growth and the approximation to them by simple curves, and frequency curves and surfaces leading to those for normal frequency. These are topics which usually receive careful mathematical treat-

[1] "*Graphic Methods for Representing Facts*," by W. C. Brinton.

ment. The author in his discussion of them suffers from a self-imposed handicap because the mathematical processes used will be considered difficult by many of those for whom the book is intended.

A great deal is being written at the present time on the subject of economic research, with special emphasis on the problem of forecasting. Efforts are being made to devise methods by means of which variations in one series may be used to forecast variations in another. To the solution of this important problem, the author [2] is making important contributions. It is the opinion of one who has worked into the field of economic research from the mathematical side, and who has confessedly brought with him his love for quantitative statement, that there is more hope of success along lines of analysis than along those whose main appeal is to the eye. A comparison between two graphs may suggest a relation between the two, yet the question as to whether this relation actually exists must be determined by mathematical analysis often of great complexity. That future progress in economic research will be made along mathematical, rather than graphical, lines is a thesis which probably would not be supported by the author or by Mr. Carl Snyder, who has written a timely introduction to the volume under consideration.

C. C. MORRIS.

*The Quantum Theory.* By FRITZ REICHE, translated by H. S. HATFIELD and H. L. BROSE. New York, E. P. Dutton and Company, 1924. 183 pages. Price $2.50.

This little book contains a systematic and compact review of the quantum theory and of its applications to thermal radiation, specific heats, optical spectra and X-rays. The simpler parts of the mathematical theory are given in detail, but the more complicated parts are simply sketched. This plan is a good one, but unfortunately the author has not always succeeded in drawing a clear sketch: for instance, the discussion of the lattice-theory of atomic heats can hardly mean much to a reader unfamiliar with the complete theory. Impartiality is carried to such a point that some readers may not realize how largely the earlier formulations of the theory have been superseded. A very valuable feature is the abundance of references, which appear to cover the literature down to 1920.

Errors of fact or of translation are scarce. We will note only that the translator seems unwilling to believe that "Impulsmoment" really means merely "moment of momentum," and falls back later upon the erroneous translation "impulse."

In the absence of a preface one cannot be sure for what class of readers the book was intended by the author. It is quite unsuited for use by a class and would hardly do even as a first introduction for a more experienced reader. It will, however, serve admirably as a good index to the quantum theory as it existed four or five years ago.

E. H. KENNARD.

[2] "The Theory of Quadrature in Economics," by K. G. Karsten, *Journal American Statistical Society*, March, 1924.

## ARTICLES IN CURRENT PERIODICALS.

**The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.**

**BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,** volume 30, nos. 5–6, May–June, 1924: "A characterization of surfaces of translation" by E. P. Lane, 231–232; "Concerning a suggested and discarded generalization of the Weierstrass factorization theorem" by L. L. Dines, 233–236; "The class number relations implicit in the Disquisitiones Arithmeticae" by E. T. Bell, 236–238; "Number of cycles of the same order in any given substitution group" by G. A. Miller, 239–246; "Algebras and their arithmetics" by L. E. Dickson, 247–257.

**JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES,** series 9, volume 3, no. 3, 1924: "Sur les intégrales multiples des variétés algébriques" by S. Lefschetz, 319–343.

**THE MESSENGER OF MATHEMATICS,** volume 53, no. 11, March, 1924: "Theta expansions useful in arithmetic" by E. T. Bell, 166–176.

**PHILOSOPHICAL MAGAZINE AND JOURNAL OF SCIENCE,** ser. 6, volume 48, no. 283, July, 1924: "On the chance of an electron being ejected photoelectrically from an atom by X-rays" by G. E. M. Jauncey, 81–88.

**PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY,** ser. 2, volume 21: "An analytical treatment of the 3-bar curve" by F. V. Morley, 140–160; "On lines of electric induction and the conformal transformations of a space of four dimensions" by H. Bateman, 256–270.

**PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES,** volume 10, no. 6, June, 1924: "A statistical discussion of sets of precise astronomical measurements, II: proper motions" by E. B. Wilson and W. J. Luyten, 228–231; "An explanation of the gaps in the distribution of the asteroids according to their periods of revolution" by E. W. Brown, 248–253; "Note on some statistical consequences of the luminosity law" by W. J. Luyten, 260–264; "On parametric representations of continuous surfaces" by B. de Kerékjártó, 267–271.—No. 7, July, 1924: "Second note: Electrodynamics in the general relativity theory" by G. Y. Rainich, 294–298.

**RENDICONTI DEL CIRCOLO MATHEMATICO DI PALERMO,** volume 48, part 1, 1924: "A generalization of evolutes" by J. L. Walsh, 23–27.

**SCIENTIFIC MONTHLY,** volume 19, no. 1, July, 1924: "The origin, nature, and influence of relativity" by G. D. Birkhoff, 18–29.—No. 2, August, 1924: "The origin, nature and influence of relativity" (concluded) by G. D. Birkhoff, 180–187.

**TOHÔKU MATHEMATICAL JOURNAL,** volume 21, 1922: "On plane algebraic curves which are invariant under a quadric Cremona transformation" by A. Emch, 310–326.

**TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY,** volume 26, no. 1, January, 1924: "An existence theorem for the characteristic number of a certain boundary value problem" by H. T. Davis, 1–16; "A theorem on the factorization of polynomials of a certain type" by L. L. Dines, 17–24; "A fundamental class of geodesics on any closed surface of genus greater than one" by H. M. Morse, 25–60; "The Hilbert integral and Mayer fields for the problem of Mayer in the calculus of variations" by G. A. Larew, 61–67; "Normal congruences and quadruply infinite systems of curves in space" by J. Douglas, 68–100; "The equivalence of certain regular transformations" by L. L. Silverman, 101–112; "MacLaurin expansion of the interpolation polynomial determined by $2n + 1$ evenly spaced points" by G. Rutledge, 113–123; "On covariants of linear algebras" by C. C. MacDuffee, 124–132; "A generalized problem in weighted approximations" by D. Jackson, 133–154.

# UNDERGRADUATE MATHEMATICS CLUBS.

**All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.**

## CLUB ACTIVITIES.

### The Mathematical Club of Adelphi College, Brooklyn, N. Y.

[*1922*, 24.]

The purpose of the club has been to promote interest in mathematics outside the classroom. During the course of its existence there have been many interesting discussions at the meetings. Such topics as the history of mathematics from early times down to modern times, women mathematicians, mathematical puzzles and tricks have been greatly favored by the members. Even the Einstein theory was discussed with interest where understanding was possible.

Meetings are held regularly at the College. At the beginning of each semester, a party is given to the new freshman members, and at the end of the year, a closing party is held. An occasional supper meeting is also held.

Dr. Joseph Bowden, professor of mathematics at Adelphi College, was the first president of the club and has held the position of honorary president ever since. The club owes a great part of its success to Dr. Bowden who has given it much of his time.

This year has been a memorable one in the history of the Adelphi College Mathematical Club for it celebrated its twenty-fifth anniversary this year. This was the most important meeting ever held by the club. Many alumni attended and the event proved successful also as a reunion. A play was given in which the activities of the club were featured. In the play, all terms used were mathematical and when these terms were applied to everyday life, the result was most unusual and interesting. The author and coach was Miss Mildred Goss '24 who was also toastmistress. Among those who made addresses were Dr. Joseph Bowden, Mr. L. L. Locke, first treasurer of the club, Miss Ruth Van Gaasbeek, president, and Miss Goss.

At the close of the entertainment, Miss Goss, in behalf of the senior members, presented to the president a gavel. Two members, who were waitresses in the play, brought in a birthday cake with twenty-five candles. Dr. Bowden, as honorary president, cut the first slice and presented it to the president, Miss Gaasbeek. Refreshments were served. Many of the older members expressed a desire to attend future meetings of the club. This proves that the anniversary meeting accomplished a double purpose, entertainment and revival of interest.

The officers of the club are: honorary president, Dr. J. Bowden; president, Miss R. Van Gaasbeek; vice-president, Miss Mildred Newman; secretary, Miss Violet Miller; treasurer, Miss Rose Brody.

(Report by Miss Violet Miller, secretary.)

### The White Mathematics Club, University of Kentucky, Lexington, Ky.

[*1923*, 336.]

The officers of the White Mathematics Club for the year 1923–1924 were: president, Professor P. P. Boyd; commissary, Professor Flora Elizabeth LeStourgeon; secretary, Professor E. L. Rees.

The following papers were presented at the meetings in 1923–1924:

October 25, 1923. "Old methods of computing" by Professor P. P. Boyd.

November 8, 1923. "Some fundamentals in the theory of integral equations" by Professor F. E. LeStourgeon.

November 22, 1923. "Involutions" by Mr. W. R. Hutcherson, Gr.

December 6, 1923. "The error function" by Professor H. H. Downing.

January 17, 1924. "History of algebraic symbolism" by Professor J. M. Davis.

February 21, 1924. "Interpretations of inverse hyperbolic and circular functions as areas" by Mr. J. C. Nixon, instructor.

March 6, 1924. "Some fallacies in the theory of probability" by Mr. M. C. Brown, Gr.

March 20, 1924. "Graphical solution of equations" by Mr. T. Andrew, instructor.

April 10, 1924. "Calculation of the date of Easter" by Professor H. H. Downing.

May 11, 1924. "The nine point circle" by Miss Helen McGurk '24.

May 23, 1924. "Intrinsic equations" by Miss L. Kuykendall '24. "Constructions of a tangent to an ellipse" by Mr. H. W. Mobley '24.

(Report by Professor E. L. Rees, secretary.)

## Denison Mathematics Club, Denison University, Granville, O.

[*1922*, 25.]

September 20, 1921. "Mathematical fallacies" by Professor R. Sheets.
September 27, 1921. "Graphical methods" by Professor F. B. Wiley.
October 25, 1921. Debate: "Resolved that freshman mathematics should be required for graduation from Denison University," affirmative, Miss Finley, Mr. Chandler; negative, Mr. Seasholes, Mr. Burke.
November 8, 1921. History of mathematics (first of a series), by Charlotte Larsen. "Desargues Theorem" by Mr. Lemon, faculty.
November 22, 1921. "Fourth dimension" by Messrs. Linebaugh, Quinn, Powell, Holt.
December 6, 1921. "History of Egyptian mathematics" by Alma Chambers. "Bible numerics" by Mary Packer.
January 17, 1922. Social meeting.
February 28, 1922. "Greek mathematics" by Mr. Robt. Case. "Infinitesimals" by Mr. Davis.
March 14, 1922. Solution of prize problems by Messrs. Bannister, Seasholes, Jones.
April 25, 1922. "The coördinate systems of the relativity theory" by Professor Befeld.
May 5, 1922. Annual banquet.
September 26, 1922. Informal talk by Professor Sheets.
October 10, 1922. "Magic Squares" by Mr. Chandler.
November 7, 1922. "Introduction to the Einstein theory" by Professor F. B. Wiley.
November 21, 1922. "Einstein theory" by Professor F. B. Wiley.
December 5, 1922. "Einstein theory" by Professor F. B. Wiley.
January 16, 1923. Social meeting.
February 13, 1923. "Radio" by Mr. Howe, faculty.
March 13, 1923. "Parallel axes" by Mr. H. B. Lemon, faculty.
April 10, 1923. "Inversions" by Professor A. B. Peckham.
April 24, 1923. "Inversions" by Anne Marshall.
May 11, 1923. Annual banquet with Professor C. T. Bumer of Ohio State University as speaker.
May 22, 1923. "Arithmetical progressions of high order" by Irene Kissling. Prize problems by George Stibitz and Mr. Bannister.
September 25, 1923. "The classification of geometries" by Professor Sheets.
October 9, 1923. "Magic Squares" by Mr. Ellis Powell, president of the club.
October 23, 1923. "Unlimited numbers" by Professor F. B. Wiley.
November 6, 1923. "Trilinear coördinates" by Mr. Donald Fitch.
November 20, 1923. "The number 10" by Miss N. Alspach. "The number" by Mr. Stibitz.
December 4, 1923. The abacus (a demonstration) by Mr. Kato, Mr. Matsuhashi.
December 18, 1923. "Mathematical fallacies" by Messrs. Bannister, Bash, Gay.
January 15, 1924. "A perpetual calendar" by Messrs. Lester Hunt, Leland Powell, Samuel Treharne.
February 5, 1924. Social hour.
February 19, 1924. "Logarithms of negative numbers" by Mr. L. Bone, Miss H. Dunlap, Mr. K. Holt.
March 18, 1924. "Rubbing elbows with infinity" by Professor F. B. Wiley.
April 29, 1924. Prize problem by Mr. Powell.
May 2, 1924. Annual banquet with Professor C. H. Yeaton of Oberlin College as speaker.
May 13, 1924. "Probability" by Professor F. B. Wiley.
May 29, 1924. Prize problem by Mr. Powell.

(Report by Professor Wiley.)

## The Newtonian Society of the State College of Washington, Pullman, Wash.

[*1922*, 27.]

The officers for the year 1923–1924 were: president, Mildred Hunt '24; secretary and treasurer, Katheryn Maloney '25; reporter to the college paper, Mildred Allgood '27. The following programs were given during the year:
October 30, 1923. "Sets of points" by Professor C. A. Isaacs. "Limits" by Mildred Hunt '24.

November 21, 1923. "First mathematical works in America" by Edna Wetherall '24. "Repeating decimals" by Mildred Allgood '27. "Mathematical publications and societies of America" by Professor Colpitts.
December 12, 1923. "Logarithms" by Mr. Irwin, instructor. "Magic Squares" by Frances Helmar '25.
January 9, 1924. "The calendar" by Miss Freeman, instructor. "Date of Easter" by Fern Bolick '26.
January 23, 1924. "$\pi$" by Robert Kennedy '26. "Trisecting an angle" by Maude Poston '24.
February 13, 1924. "Number Systems of Indians" by Mildred Hunt '24. "Perfect numbers" by Ruth Himmelsback '27.
February 27, 1924. "Determinants" by Mr. Akey, instructor.
March 12, 1924. "Oneonu" by Katheryn Maloney '25. "Series" by Flossie Folsom, Gr.
March 26, 1924. "Geometric fallacies" by Miss Hunt '24, Miss Poston '24, Miss Wetherall '24.
(Report by Professor C. A. Isaacs.)

## The Mathematics Club of the University of Oregon, Eugene, Oregon.

[*1918*, 315.]

October 23, 1923. "Quaint old mathematical books" by Professor E. E. DeCou.
November 27, 1923. Ciphering match at the black board.
January 22, 1924. "Divergent series" by Dr. L. L. Small.
March 4, 1924. Joint meeting with Philosophy Club. "Mathematical conception of infinity" by Professor W. E. Milne.
April 9, 1924. Social.
May 13, 1924. "Need of mathematics in biology" by Oscar Richards, Gr. Joint meeting with Biology Club.
June 6, 1924. Initiation of new members and installation of new officers.

The officers for the year 1924–1925 are: president, Vera Hughes; vice-president, Sylvia Veatch; secretary, Leola Craig; treasurer, Roland Humphreys; historian, Dorothy Akin.

The officers for the year 1923–1924 were: president, Wave Lesley; vice-president, Sylvia Veatch; secretary, Willa Loomis; treasurer, Vera Hughes.

(Report by Miss Leola Craig, secretary.)

## The Mathematical Club, Harvard University, Cambridge, Mass.

[*1923*, 273.]

The following programs were given before the Harvard Mathematical Club for the year 1923–1924:

October 9, 1923. "Analytic functions of several complex variables" by Professor W. F. Osgood.
October 23, 1923. "An irregular expansion problem" by Mr. L. E. Ward.
November 6, 1923. "The representation of functions by formulas of interpolation" by Mr. K. W. Halbert.
November 20, 1923. "The inversion of elliptic integrals" by Mr. D. V. Widder.
December 5, 1923. "Some theorems on Fourier series" by Professor O. D. Kellogg.
December 19, 1923. "The rigidity of oval surfaces" by Mr. M. M. Slotnick.
January 15, 1924. "Tropistic geometry" by Dr. H. M. Sheffer.
February 13, 1924. "Collineations of 3-space" by Mr. A. J. Cook.
February 26, 1924. "The foundations of geometry" by Professor G. D. Birkhoff.
March 11, 1924. "The convergence of Fourier series" by Mr. M. H. Stone.
March 25, 1924. "Flow and stability" by Mr. B. O. Koopman.
April 8, 1924. "Existence theorems for differential equations" by Dr. Philip Franklin.
April 29, 1924. "The oscillations of harmonic functions" by Mr. F. W. Perkins.
May 13, 1924. "Quantum theory" by Professor E. C. Kemble.
May 27, 1924. "Cumulative numerical integration" by Professor H. N. Davis.
(Report by Professor O. D. Kellogg.)

# PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

**Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.**

## PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

**3098. Proposed by JEAN WINSTON, University of Cincinnati.**

Find the involutes of the parabola $y = x^2$.

**3099. Proposed by S. A. COREY, Des Moines, Iowa.**

Let $H$, $I$, $J$, and $K$ be the complex quaternion units, $\frac{1}{2}[1 - \theta(i + k)]$, $\frac{1}{2}[1 + j + \theta(i - k)]$, $\frac{1}{2}[1 + j - \theta(i - k)]$, $\frac{1}{2}[1 - j + \theta(i - k)]$, respectively, which have the "multiplication table"

| | $H$ | $I$ | $J$ | $K$ |
|---|---|---|---|---|
| $H$ | $H$ | $I$ | $H$ | $-I$ |
| $I$ | $H$ | $I$ | $-H$ | $I$ |
| $J$ | $J$ | $-K$ | $J$ | $K$ |
| $K$ | $-J$ | $K$ | $J$ | $K$. |

Prove that there exists a set of four matrices, involving no imaginaries, which have the same "multiplication table."

**3100. Proposed by H. E. TREFETHEN, Colby College.**

Show that the segment between the axes tangent to the astroid at any point and the radius of the fixed circle to its point of contact with the generating circle bisect each other.

## SOLUTIONS.

**3050 [1924, 49]. Proposed by C. N. MILLS, State Normal School, Aberdeen, South Dakota.**

Eliminate $x$, $y$, $z$ from the equations

$$x^4/a^{5/4}b^{3/4} + y^4/a^{3/4}b^{5/4} = mz^2,$$
$$x^{3/2}/(az)^{1/4} = x + y = y^{3/2}/(bz)^{1/4},$$

and show that, if $ab > 0$, $m$ cannot be less than $2^9$.

### SOLUTION BY HAZEL E. SCHOONMAKER.

From the second equations we have $x^6 = ay^6/b$, from which we obtain $x$. This value of $x$ inserted in the first equation gives

$$z = \pm \frac{(a^{1/6} + b^{1/6})^{1/2}y^2}{(a^9b^{17}m^{12})^{1/24}}.$$

Inserting the values of $x$ and $z$ in $x + y = y^{3/2}/(bz)^{1/4}$ we obtain

$$y^4(a^{1/6} + b^{1/6})^{9/2} = a^{3/8}b^{3/8}m^{1/2}y^4,$$

whence

$$m = \left(c + \frac{1}{c}\right)^9, \qquad \left(\frac{a}{b}\right)^{1/12} = c.$$

Assume $c + 1/c < 2$. Then $c^2 + 1 < 2c$ or $(c - 1)^2 < 0$. This is impossible if $c$ is real. Hence $m \geqq 2^9$.

NOTE BY EDITORS: In this proof it is assumed that $y$ is not zero. If $y$ is zero, $x$ and $z$ also vanish, and $m$ may have any value; thus in this case the theorem in the problem is not true.

Also solved by J. A. BULLARD, A. C. CLARK, F. S. GACHET, J. S. GEORGES, A. M. HARDING, G. A. KREINS, A. PELLETIER and E. E. WHITFORD.

**3052 [1924, 49]. Proposed by DR. JOSEF LEWAMDONSKI, Pfaffsatten, Austria.**

The ellipse whose parametric equations are

$$x = a\cos\varphi, \qquad y = b\sin\varphi,$$

intersects the conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

If $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ are the eccentric angles of the four points of intersection, prove that

$$\tan\tfrac{1}{4}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = abB/(a^2A - b^2C).$$

SOLUTION BY O. J. PETERSON, University of Michigan.

Denote by $Q_1$, $Q_2$, $Q_3$, $Q_4$ the four points on the major auxiliary circle of the ellipse whose polar angles are the eccentric angles of the four points of intersection of the ellipse and conic.

Replace $y$ by $(b/a)\,y$ in the equations of the ellipse and conic. Then the ellipse is replaced by its major auxiliary circle $x^2 + y^2 = a^2$, and the given conic is replaced by the conic

$$S \equiv a^2Ax^2 + abBxy + b^2Cy^2 + a^2Dx + abEy + a^2F = 0.$$

The points of intersection of the circle and $S = 0$ are the points $Q_1$, $Q_2$, $Q_3$, $Q_4$. If $\alpha$ be the angle which an axis of symmetry of $S = 0$ makes with the $x$-axis,

$$\tan 2\alpha = \frac{abB}{a^2A - b^2C}.$$

Now rotate the axes through the angle $\alpha$. Let the coördinates of $Q_1$ referred to the new axes be $x_1'$, $y_1'$, and its polar angle $\varphi_1'$; similarly for $Q_2$, $Q_3$, $Q_4$. The slope of the chord $Q_1Q_2$ equals

$$\frac{y_1' - y_2'}{x_1' - x_2'} = -\frac{x_1' + x_2'}{y_1' + y_2'} = -\frac{\cos\varphi_1' + \cos\varphi_2'}{\sin\varphi_1' + \sin\varphi_2'} = -\cot\frac{\varphi_1' + \varphi_2'}{2}.$$

and the slope of $Q_3Q_4$ equals $-\cot\dfrac{\varphi_3' + \varphi_4'}{2}$.

Since $Q_1Q_2$ and $Q_3Q_4$ are opposite common chords of a circle and a conic with an axis parallel to the $x'$-axis, the angles which these chords make with the $x'$-axis are supplementary, and hence the slope of $Q_1Q_2$ is the negative of the slope of $Q_3Q_4$; that is,

$$-\cot\frac{\varphi_1' + \varphi_2'}{2} = +\cot\frac{\varphi_3' + \varphi_4'}{2}.$$

and

$$\frac{\varphi_1' + \varphi_2'}{2} + \frac{\varphi_3' + \varphi_4'}{2} = n\pi.$$

Since $\varphi_1 = \varphi_1' + \alpha$, $\varphi_2 = \varphi_2' + \alpha$, $\varphi_3 = \varphi_3' + \alpha$, $\varphi_4 = \varphi_4' + \alpha$,

$$\frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{2} = 2\alpha + n\pi,$$

and finally,

$$\tan\frac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = \tan 2\alpha = \frac{abB}{a^2A - b^2C}.$$

NOTE BY THE EDITORS: A proof of the theorem regarding the common chords of a circle and a conic is given in the note on the solution of Problem 2990 (*1924*, 51).

The part of the solution following the derivation of $\tan 2\alpha$ may be shortened by a direct computation of the angles from a figure. Thus: the perpendiculars from the center to the chords $Q_1Q_2$ and $Q_3Q_4$ of the circle have polar angles which may be written $(\varphi_1 + \varphi_2)/2$ and $(\varphi_3 + \varphi_4)/2$, neglecting multiples of $\pi$. By the theorem cited above a line parallel to an axis of the conic bisects the angle between the pair of chords, also the angle between the corresponding

pair of perpendiculars. Hence we may write, neglecting multiples of $\pi$,

$$\frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{2} = 2\alpha,$$

and the desired result follows at once.

Also solved by A. BOGARD, J. A. BULLARD, WILLIAM HOOVER, C. K. ROBBINS, HAZEL E. SCHOONMAKER and the PROPOSER.

**3055 [1924, 101]. Proposed by J. L. RILEY, Tarleton Station, Texas.**

Given the non-intersecting circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$
$$x^2 + y^2 + a_2x + b_2y + c_2 = 0;$$

it is required to find the four common tangents.

## SOLUTION BY ROSCOE WOODS, University of Iowa.

Let $O_1$, $O_2$ be the centers of the circles $C_1$, $C_2$ whose radii are $r_1$, $r_2$ respectively. The two points $S_1$, $S_2$ which divide the line $O_1O_2$ internally and externally in the ratio $r_1 : r_2$ are called the centers of similitude. Two of the common tangents to the two circles $C_1$, $C_2$ pass through each center of similitude.

The coördinates of $O_1$, $O_2$, $S_1$, $S_2$ are $(-a_1, -b_1)$, $(-a_2, -b_2)$ [1] and $\left(-\frac{a_1r_2 \pm a_2r_1}{r_2 \pm r_1}, -\frac{b_1r_2 \pm b_2r_1}{r_2 \pm r_1}\right)$ respectively. The values of $r_1$ and $r_2$ in terms of $a_1$, $b_1$, etc., are $\sqrt{a_1^2 + b_1^2 - c_1}$, $\sqrt{a_2^2 + b_2^2 - c_2}$ respectively. The coefficients $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$ are assumed to be real.

The equation of a line through $S_1$ or $S_2$ with a variable slope $m$ is

$$(y - mx)(r_2 \pm r_1) + b_1r_2 \pm b_2r_1 - m(a_1r_2 \pm a_2r_1) = 0.^2 \tag{1}$$

The condition that this line be tangent to one of the circles, $C_1$ say, is that the distance from this line to the point $O_1(-a_1, -b_1)$ be $r_1$. This condition when simplified is

$$\pm\sqrt{1 + m^2}(r_2 \pm r_1) = m(a_1 - a_2) + (b_2 - b_1). \tag{2}$$

(The ambiguous signs are independent.)

If (2) is rationalized, a quadratic in $m$ results. The roots of this quadratic equation are the slopes of the pair of common tangents to the circles $C_1$, $C_2$ through each center of similitude, that is, a pair of tangents is given by a choice of the ambiguous sign. These roots are

$$m = \frac{(a_1 - a_2)(b_1 - b_2) \mp (r_2 \pm r_1)\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 - (r_2 \pm r_1)^2}}{(a_1 - a_2)^2 - (r_2 \pm r_1)^2}. \tag{3}$$

The four values of $m$ obtained from (3) set in (1) give the equations of the four common tangents to $C_1$, $C_2$ regardless of the character of the radical. Since these expressions are cumbersome, they are not inserted at length.

In order to discuss the various cases that are possible, note that the sum of the first two terms under the radical in (3) is the square of the distance $O_1O_2$. Choose the plus sign. The pair of tangents is real, coincident, or imaginary according as the expression under the radical is positive, zero or negative. This expression is positive when one circle lies wholly without the other; is zero when the circles are tangent externally; and is negative when the circles intersect in two real points or one lies wholly within the other. Similarly, when the minus sign is chosen, the pair of tangents is real, coincident, or imaginary according as the expression under the radical sign is positive, zero or negative. This expression is positive when one circle lies wholly without the other or when the two circles intersect in two real points; is zero when the two circles are tangent internally, and is negative when one circle lies wholly within the other. Hence

[1] Note the slight change in notation, $2a_1$ is written for $a_1$, etc.

[2] Assume $r_1 \neq r_2$ for the present. The case $r_1 = r_2$ is discussed later.

all the tangents are real when one circle lies wholly without the other, one pair real and the other pair imaginary when the circles intersect in two real points, and both pairs are imaginary when one circle lies wholly within the other.

When $r_1 = r_2$, the center of similitude $S_2$ (given by the minus sign) is infinitely distant. If we refer to (2) with the minus sign before $r_1$ just one value of $m$ is given, *i.e.*, $m = (b_2 - b_1)/(a_2 - a_1)$ as it should be. In case the circles coincide, the slope is indeterminate unless the direction along which the center $O_1$ approaches $O_2$ is given.

If $r_1 \neq r_2$ and the circles are concentric, (2) shows that there are only two values of $m$, *i.e.*, $m = \pm\sqrt{-1}$. In this case all four common tangents pass through the common center $O_1$ and are coincident in pairs. They are the isotropic or minimal lines through $O_1$, namely

$$(y + b_1) \pm i(x + a_1) = 0 \qquad \text{where} \qquad i = \sqrt{-1}. \tag{4}$$

Finally, it is to be noted that if the radii of both circles are unreal (that is, pure imaginary numbers as is the case since the coefficients $a_1$, etc., are assumed to be real) all four tangents can be real. If one radius is real and the other unreal all four tangents are unreal.

Also solved by C. S. ATCHISON, WILLIAM HOOVER, A. PELLETIER, W. B. PIERCE, HAZEL SCHOONMAKER and J. K. WHITTEMORE.

**3058 [1924, 101]. Proposed by LOUIS WEISNER, University of Rochester.**

If $n$ is any integer greater than 1, the number of integers less than $n$ and prime to $n$ of the form $c + xd$ is $\phi(n)/\phi(d)$, where $d$ is a divisor of $n$ and $c$ is an integer less than $d$ and prime to $d$.

SOLUTION BY L. C. MATHEWSON, Dartmouth College.

Let $b_1 = 1, b_2, b_3, \cdots, b_{\phi(n)}$ be the positive integers less than $n$ and prime to $n$, and let $c_1 = 1, c_2, \cdots, c_{\phi(d)}$ be the positive integers less than $d$ and prime to $d$.

Separate the $b$'s into sets placing all those of the form $c_1 + xd$ into the first set; all those (if any) of the form $c_2 + xd$ into the second set; etc. No $b$ will be in two different sets, for dividing any one $b$ by $d$ could not give two different $c$'s for remainders.

We shall next show that there are the same number of $b$'s in each set. Since $c_i$ is prime to $d$, there exists by number theory an integer $k$ such that $c_i k \equiv c_j$, mod $d$. Multiplying the integers $\equiv c_i$ by $k$ would give the same number of distinct integers $\equiv c_j$. Hence there are at least as many $b$'s in the $j$th set as in the $i$th set. Similarly, there are as many in the $i$th set as in the $j$th set. Since there are $\phi(n)$ $b$'s and $\phi(d)$ sets, the number in any one set is $\phi(n)/\phi(d)$.

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## NOTES AND NEWS.

**Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.**

At the meeting of the British Association for the Advancement of Science at Toronto in August, 1924, Professor HORACE LAMB was elected president, for the meeting to be held in Southampton in 1925. Sir W. H. BRAGG, as president of the section of Mathematical and Physical Sciences, delivered an address on *Crystal Structure*. On the occasion of the meeting, the University of Toronto conferred the honorary degree of doctor of science on Sir ERNEST RUTHERFORD, retiring president of the Association.

Professor J. C. MCLENNAN, of the University of Toronto, has been elected president of the Royal Society of Canada.

Professor R. A. MILLIKAN, of the California Institute of Technology, has received the honorary degree of doctor of science from Trinity College, Dublin.

Professor A. S. EDDINGTON, of Cambridge University, has been elected an honorary member of the American Astronomical Society.

Professor M. I. PUPIN, of Columbia University, has received the honorary degree of doctor of science from Princeton University.

The private scientific library of the late Professor A. G. WEBSTER, of Clark University, has been purchased by the Riverbank Laboratories, Geneva, Illinois, and is now housed there as a separate collection.

At the meeting of the International Mathematical Congress held during August in Toronto, Professor S. PINCHERLE, of the University of Bologna, was elected president of the International Mathematical Union, and Professor J. C. FIELDS, of the University of Toronto, was elected president of the Congress for its next meeting.

The third Pan-American Scientific Congress will be held at Lima, Peru, beginning December 20, 1924. The officers of Section II (Mathematical and Physical Sciences) are Vice-Admiral M. M. CARBAJAL, chairman, and Professor J. R. DE LA PUENTE, secretary. The chairman of Subsection I (Pure Mathematics, Rational Mechanics, Mathematical Physics) is Professor E. VILLARÁN.

At Bryn Mawr College, Professor CHARLOTTE A. SCOTT has retired; she is succeeded as head of the department of Mathematics by Professor ANNA J. PELL. Dr. D. V. WIDDER has been appointed associate in mathematics.

Dean E. G. BILL, of Dartmouth College, is serving as chairman of a commission appointed by the College Entrance Examination Board to consider the desirability of psychological tests being given by the Board.

Professor S. LEFSCHETZ, who is on leave of absence from the University of Kansas, has been appointed visiting professor of mathematics at Princeton University for the academic year 1924–1925.

Dr. MAYME I. LOGSDON, of the University of Chicago, has been awarded a foreign fellowship by the General Education Board for study abroad during the year 1925–26. She will spend the time in study in Italian universities. She will sail for Italy in June, 1925.

The officers of the Association now and then receive requests for a possible donation of mathematical books for libraries in the smaller institutions where funds are not available for mathematical books. It is more than probable that individual or institutional members of the Association may have duplicate copies of mathematical books which they would be glad to donate to such institutions if they only knew where the need exists. Any such information will be acted upon if sent to the Secretary of the Association.

The composition for the first Carus Monograph is under way. It is expected that it will come from the press early in January, 1925.

All members of the Association should receive soon a communication from the Joint Committee on Membership of the Association and the Society, indicating the steps which are being taken to secure the membership in one or the other, but especially in *both*, of these organizations. An attempt is made to reach directly every person who is teaching collegiate mathematics and who is not now

supporting both the Society and the Association. Every member of the Association is requested to assist this Committee by his personal influence among non-members and by giving to the Committee any information which may be of use in forwarding this work. Please address the Secretary, Professor W. D. CAIRNS, Oberlin, Ohio, if you have any suggestions to make.

The Commission for Relief in Belgium Educational Foundation announces that a limited number of American Graduate Fellowships for study in Belgium during the academic year 1925–1926 will be awarded by April 1, 1925. Preference in selection is given to applicants between the ages of twenty-five and thirty-three who are unmarried and who intend to take up teaching or research as a profession. Not more than six fellowships will be awarded for 1925–1926; fellowships may be held in any one of twenty subjects, of which mathematics is one. Applications must reach the Committee (Fellowship Committee, C-R-B Educational Foundation, Inc., 42 Broadway, New York City) by February 15, 1925.

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